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Conjunction and Quasi Conjunction of Conditionals in Coherence-Based Probabilistic Nonmonotonic Reasoning

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Outline:

- Coherence-based probabilistic reasoning;

- Probabilistic results on Quasi Conjunction (Adams) and QAND rule (Dubois & Prade);

- Conjunction of conditionals (S. Kaufmann 2009), properties, CONJUNCTION rule;

- A comparison with Quasi Conjunction;

- Particular cases: biconditionals, AND, OR, CM, CUT rules;

- Generalization and degradation of inference rules: the case of OR rule;

- An apparent paradox on conjunction, with an example.

Coherence-based probabilistic reasoning

Nonmonotonic reasoning has been studied by many, symbolic or numerical, formalisms.

The inferential process is developed by applying inference rules to conditional knowledge bases (i.e., sets of conditional assertions, like "if A then B", which we represent by $B|A$).

Conditional assertions may have exceptions and conditional knowledge bases may be arbitrary.

To quantify uncertainty and to analyze the degradation of inference rules, we need numerical formalisms (*flexible, general and with a clear rationale*).

Coherence-based probabilistic reasoning:
- is flexible (we can consider arbitrary conditional knowledge bases);

- is general (we can directly assess conditional probabilities);
- has a clear rationale, represented by a very intuitive axiom: the coherence principle of B. de Finetti.

Basic notions on coherence

Events are two-valued logical entities (*true*, or *false*; in numerical terms: 1, or 0), described by propositions.

A conditional event $E|H$ is a three-valued logical entity, *true*, or *false*, or *void*, according to whether E and H are true, or E is false and H is true, or H is false.

We can measure a *degree of belief* by the *betting scheme*:
If you assess $P(E) = p$, then you are willing to

pay p (resp., to receive p), by receiving (resp., by paying) the amount E, equal to 1, or 0, according to whether E is true, or false.

Your belief should be *coherent*, that is:
you should want to avoid transactions that would surely yield for you a net loss, no matter what the value of E happens to be.

For the conditional assessment $P(E|H) = p$, the bet is *valid* if H is *true* and is *called off* if H is *false*.
You pay $p = P(E|H)$, by receiving an amount equal to the indicator $E|H$, defined as

$$E|H = \begin{cases} 1, & EH \; true \\ 0, & E^c H \; true \\ p, & H^c \; true \end{cases}$$

Then: $E|H = EH + pH^c \in \{1, 0, p\}$.

By *linearity* of prevision, we obtain

$$P(E|H) = p = P(EH) + pP(H^c)\,;$$

hence: $P(EH) = P(E|H)P(H)$, and

$$P(E|H) = \frac{P(EH)}{P(H)}, \quad (\text{if } P(H) > 0).$$

If $P(H) = 0$, coherence just requires $0 \leq p \leq 1$.

Coherence implies all the classical properties of conditional probability, such as

$$P(AB|H) = P(A|H)P(B|AH), \ (AH \neq \emptyset);$$

$$P(A \vee B|H) = P(A|H) + P(B|H), \ (AB = \emptyset).$$

For a conditional random quantity $X|H$, with the betting scheme, if we assess $\mathbb{P}(X|H) = \mu$ then we pay (we receive) μ by receiving (by paying) the amount

$$X|H = \begin{cases} X, & H \\ \mu, & H^c \end{cases} = XH + \mu H^c,$$

and, by linearity of prevision, it holds

$$\mathbb{P}(X|H) = \mu = \mathbb{P}(XH + \mu H^c) = \mathbb{P}(XH) + \mu P(H^c),$$

from which it follows: $\mathbb{P}(XH) = P(H)\mathbb{P}(X|H)$.

More in general, by linearity of prevision, we have:

$$\mathbb{P}(XH|K) = P(H|K)\mathbb{P}(X|HK)\,.$$

The checking for coherence amounts to study the solvability (of a finite sequence) of suitable linear systems.

Conditional probabilistic logic of Adams and Quasi Conjunction

In the setting of coherence (Gilio 2002, Gilio 2011) we can define with full generality the notion of p-consistency and that of p-entailment, denoted $\Rightarrow_p$.

Definition 1. The family $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1,\ldots,n\}$, is *p-consistent* iff, for every set of lower bounds $\{\alpha_i, i = 1,\ldots,n\}$, with $\alpha_i \in [0,1)$, there exists a coherent probability assessment $\{p_i, i = 1,\ldots,n\}$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, such that $p_i \geq \alpha_i, i = 1,\ldots,n$.

Definition 2. A p-consistent family

$$\mathcal{F}_n = \{E_i|H_i\,,\ i = 1,\ldots,n\}\,,$$

p-entails the conditional $B|A$ if and only if there exists a subset $\mathcal{S} = \{E_i|H_i,\ i \in \Gamma\}$ of $\mathcal{F}_n$, with $\Gamma \subseteq \{1,\ldots,n\}$, such that, for every $\alpha \in [0,1)$, *there exists* a set of lower bounds $\{\alpha_i, i \in \Gamma\}$, with $\alpha_i \in [0,1)$, such that for all coherent probability assessments $\{z, p_i, i \in \Gamma\}$ defined on $\{B|A\} \cup \mathcal{S}$, with $z = P(B|A)$ and $p_i = P(E_i|H_i)$, it holds that

$$P(E_i|H_i) \geq \alpha_i\,,\ \forall\, i \in \Gamma \ \Rightarrow\ P(B|A) \geq \alpha\,.$$

A basic notion in the work of Adams is the quasi conjunction of conditionals, defined as

$$\mathcal{C}(A|H,B|K) = (AH \vee H^c) \wedge (BK \vee K^c)|(H \vee K) =$$

$$= (AHBK \vee H^cBK \vee AHK^c)|(H \vee K)\,.$$

As it can be verified (Gilio 2004), the extension $\gamma = P[\mathcal{C}(A|H,B|K)]$ of the assessment (x,y) on $\{A|H,B|K\}$, with A,H,B,K logically independent, is coherent if and only if: $\gamma' \leq \gamma \leq \gamma''$, where

$$\gamma' = \max(x+y-1,0)\,,$$
$$(Lukasiewicz\ t-norm)$$

$$\gamma'' = \begin{cases} \frac{x+y-2xy}{1-xy}, & (x,y) \neq (1,1), \\ 1, & (x,y) = (1,1). \end{cases}$$
$$(Hamacher\ t-conorm)$$

Quasi conjunction for a family of conditional events $\mathcal{F}_n = \{E_1|H_1,\ldots,E_n|H_n\}$ is defined as

$$\mathcal{C}(\mathcal{F}_n) = \bigwedge_{i=1}^{n} (E_iH_i \vee H_i^c)\Big|(\bigvee_{i=1}^{n} H_i)\,,$$

and is associative; that is, given any partition $\{\mathcal{F}', \mathcal{F}''\}$ of $\mathcal{F}_n$, it holds that

$$\mathcal{C}(\mathcal{F}_n) = \mathcal{C}(\mathcal{F}' \cup \mathcal{F}'') = \mathcal{C}[\mathcal{C}(\mathcal{F}'), \mathcal{C}(\mathcal{F}'')] \,.$$

The extension $\gamma = P[\mathcal{C}(\mathcal{F}_n)]$ of the assessment $\mathcal{P}_n = (x_1, \ldots, x_n)$ on $\mathcal{F}_n$, with $E_1, H_1, \ldots, E_n, H_n$ logically independent, is coherent if and only if $\gamma' \leq \gamma \leq \gamma''$, where

$$\gamma' = T_L(x_1, \ldots, x_n) \,, \quad \gamma'' = S_0^H(x_1, \ldots, x_n) \,;$$

(T_L = Lukasiewicz t-norm; S_0^H = Hamacher t-conorm, with parameter $\lambda = 0$).
(Gilio & Sanfilippo 2010)

Conditional objects

Quasi conjunction also plays a relevant role in (Dubois & Prade 1994), where it is proposed the following $QAND\, rule$:

$$\mathcal{KB} \quad \text{entails} \quad \mathcal{C}(\mathcal{KB}) \,.$$

Probabilistic results on QAND rule and p-entailment (Gilio & Sanfilippo 2010, 2011)

Theorem 1. Given a p-consistent family of conditional events $\mathcal{F}_n$, for every nonempty subfamily $\mathcal{S} = \{E_i|H_i, i = 1, \ldots, s\} \subseteq \mathcal{F}_n$, we have

$$\mathcal{F}_n \Rightarrow_p \mathcal{C}(\mathcal{S})\,. \tag{1}$$

Theorem 2. Given a p-consistent family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and a conditional event $E|H$, the following assertions are equivalent:

1. $\mathcal{F}_n$ p-entails $E|H$;

2. The assessment $\mathcal{P} = (1, \ldots, 1, z)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1, i = 1, \ldots, n$, $P(E|H) = z$, is coherent iff $z = 1$;

3. The assessment $\mathcal{P} = (1, \ldots, 1, 0)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1, i = 1, \ldots, n$, $P(E|H) = 0$, is not coherent;

4. Either there exists a nonempty $\mathcal{S} \subseteq \mathcal{F}_n$ such that $\mathcal{C}(\mathcal{S}) \subseteq E|H$, or $H \subseteq E$.

5. There exists a nonempty subset $\mathcal{S} \subseteq \mathcal{F}_n$ such that $\mathcal{C}(\mathcal{S}) \Rightarrow_p E|H$.

With any pair $(\mathcal{F}_n, E|H)$ we can associate the class

$$\mathcal{K} = \{\mathcal{S} : \mathcal{C}(\mathcal{S}) \subseteq E|H \, , \, \emptyset \subset \mathcal{S} \subseteq \mathcal{F}_n\} \, ,$$

where $\subseteq$ is the inclusion relation of Goodman & Nguyen.

If not empty, the class $\mathcal{K}$ (among other properties) satisfies:

1. is *additive*:

$$\mathcal{S} \in \mathcal{K} \, , \; \mathcal{U} \in \mathcal{K} \; \Rightarrow \; \mathcal{S} \cup \mathcal{U} \in \mathcal{K} \, .$$

2. has a *greatest element* $\mathcal{S}^*$.

$\mathcal{K}$ can be determined by an algorithm which:
- checks p-consistency and p-entailment;
- computes the greatest element (if any), $\mathcal{S}^*$.

Conjunction of conditionals

In (S. Kaufmann 2009) an infinite procedure (Stalnaker Bernoulli model) is proposed to compute the probability of $A \to C$, by proving that:

$$P(A \to C) = \frac{P(AC)}{P(A)} = P(C|A)\,.$$

Then, the truth values of $A \to C$ are defined (like conditional events) as:

$$V(A \to C) = \begin{cases} 1, & AC \;\; true \\ 0, & AC^c \;\; true \\ P(C|A), & A^c \;\; true \end{cases}$$

by obtaining:
$\mathbb{P}[V(A \to C)] = P(A \to C) = P(C|A)$.

Moreover, assuming $P(A \vee C) > 0$, Kaufmann obtains for the conjunction of $A \to B$ and $C \to D$ the formula

$$P[(A \to B) \wedge (C \to D)] =$$

$$= \frac{P(ABCD)+P(B|A)P(A^cCD)+P(D|C)P(ABC^c)}{P(A \vee C)}.$$

Based on this result, Kaufmann suggests a natural way of defining the values of conjoined conditionals.

In the setting of coherence, the results of Kaufmann can be obtained (and generalized) in a direct and simpler way.

A basic aspect: we can manage without problems the case of *conditioning events with zero probability* (e.g., $P(A \vee C) = 0$) and, by starting with the assessment $P(B|A) = x,\ P(D|C) = y$, we can determine the exact lower and upper bounds, z', z'', on $z = \mathbb{P}[(A \to B) \wedge (C \to D)]$.

Notice that, as we will see, the conjunction $(A \to B) \wedge (C \to D)$ is a *conditional random quantity*; hence, we speak of *previsions* (and not probabilities) of conjoined conditionals.

Some logical and probabilistic remarks

If we consider the conjunction AB, or the conjunction $(A|H) \wedge (B|H) = AB|H$, where $H \neq \emptyset$, for the indicators it holds

$$AB = min\{A, B\} = A \cdot B \in \{0, 1\},$$

and, *conditionally on H being true*,

$$AB|H = min\{A|H, B|H\} = (A|H) \cdot (B|H) \in \{0, 1\}.$$

Then, if we assess $P(AB|H) = z$, we can write

$$(A|H) \wedge (B|H) = AB|H = \begin{cases} min\{A|H, B|H\}, & H \\ z, & H^c \end{cases}$$

$$= min\{A|H, B|H\} \cdot H + z \cdot H^c =$$

$$= min\{A|H, B|H\} \,|\, H = min\{A|H, B|H\} \,|\, (H \vee H).$$

Based on the previous formula, we define in general the conjunction of $A|H$ and $B|K$ as

$$(A|H) \wedge (B|K) = min\{A|H, B|K\} \,|\, (H \vee K).$$

If you assess $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(A|H) \wedge (B|K) \,|\, (H \vee K)] = z$, then, with the betting scheme, you agree to pay the amount z by receiving the amount

$$1 \cdot AHBK + x \cdot H^c BK + y \cdot AHK^c \in \{0, x, y, 1\}\,,$$

if $H \vee K$ is $true$, or the amount z if the bet is *called off* $(H \vee K \; false)$.

Then, *operatively*, $(A|H) \wedge (B|K)$ can be *represented* as: $(A|H) \wedge (B|K) =$

$$= 1 \cdot AHBK + x \cdot H^c BK + y \cdot AHK^c + z \cdot H^c K^c =$$

$$= (AHBK + x\, H^c BK + y\, AHK^c) \,|\, (H \vee K)\,,$$

and, by *linearity* of prevision, it follows

$$\mathbb{P}[(A|H) \wedge (B|K)] = z = P[AHBK \,|\, (H \vee K)] +$$

$$+ xP[H^c BK \,|\, (H \vee K)] + yP[AHK^c \,|\, (H \vee K)]\,.$$

Then, assuming $P(H \vee K) > 0$, it follows the result of Kaufmann

$$z = \mathbb{P}[(A|H) \wedge (B|K)] =$$

$$= \frac{P(AHBK)+P(A|H)P(H^cBK)+P(B|K)P(AHK^c)}{P(H \vee K)} .$$

Lower and upper bounds for $(A|H) \wedge (B|K)$

We assume A, H, B, K *logically independent* (in case of *logical dependencies*, lower bounds may *increase* and upper bounds may *decrease*).

We recall that the extension $z = P(AB|H)$ of the assessment (x, y) on $\{A|H, B|H\}$, with A, B, H logically independent, is coherent if and only if $z' \leq z \leq z''$, where

$$z' = \max\{x + y - 1, 0\} , \quad z'' = \min\{x, y\} .$$

In particular:
$P(AB|H) \leq P(A|H) , \quad P(AB|H) \leq P(B|H).$

The same result holds for $(A|H) \wedge (B|K)$ *!*
(Gilio & Sanfilippo 2011, *working paper*)

Given the assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if

$$\max\{x + y - 1, 0\} \ \leq \ z \ \leq \ \min\{x, y\}\,.$$

As we can see: $\mathbb{P}[(A|H) \wedge (B|K)] \leq P(A|H)$ and $\mathbb{P}[(A|H) \wedge (B|K)] \leq P(B|H)$;
while, for quasi conjunction it may be:
$P[\mathcal{C}(A|H, B|K)] \geq x\,, \ \ P[\mathcal{C}(A|H, B|K)] \geq y\,.$

Notice that: $x \to 1, y \to 1 \Rightarrow z \to 1$; that is "if the probabilities of $A|H$ and $B|K$ are *high*, then the *prevision* of $(A|H) \wedge (B|K)$ is *high*".

Then, *we can say that*:

$$\{A|H, B|K\} \quad \text{p-entails} \quad (A|H) \wedge (B|K)\,.$$

(CONJUNCTION rule)

A comparison with quasi conjunction

We have

$\mathcal{C}(A|H, B|K) = (AHBK + H^cBK + AHK^c) \,|\, (H \vee K)$;

moreover, defining $P(A|H) = x$, $P(B|K) = y$, we have: $(A|H) \wedge (B|K) =$

$$= (AHBK + xH^cBK + yAHK^c) \,|\, (H \vee K) \,.$$

Then: $\mathcal{C}(A|H, B|K) - (A|H) \wedge (B|K) =$

$= [(1-x)H^cBK + (1-y)AHK^c)] \,|\, (H \vee K) \geq 0$,

and, by linearity of prevision, it follows

$$\mathbb{P}[\mathcal{C}(A|H, B|K) - (A|H) \wedge (B|K)] =$$

$= P[\mathcal{C}(A|H, B|K)] - \mathbb{P}[(A|H) \wedge (B|K)] = \gamma - z \geq 0 \,.$

Therefore: $(A|H) \wedge (B|K) \;\Rightarrow_p\; \mathcal{C}(A|H, B|K)$;

moreover:

$x \simeq 1 \,,\; y \simeq 1 \;\Rightarrow\; (A|H) \wedge (B|K) \simeq \mathcal{C}(A|H, B|K) \,.$

A particular case: $H = B$, $K = A$.
(*biconditional event* $A \dashv\vdash B$, or *defective biconditional* (Gauffroy and Barrouillet, 2009), discussed in a short note by Andy Fugard)

Let be given a coherent assessment (x, y, z) on $\{A|B, B|A, AB \,|\, (A \vee B)\}$. We have

$$AB \,|\, (A \vee B) = AB + zA^cB^c\,,$$

$$(A|B) \wedge (B|A) = AB + xB^cAB + yA^cAB + zA^cB^c =$$

$$= AB + zA^cB^c = AB \,|\, (A \vee B) =$$

$$= (AB \vee B^c) \wedge (BA \vee A^c) \,|\, (A \vee B) = \mathcal{C}(A|B, B|A)\,.$$

Hence: $(A|B) \wedge (B|A) = \mathcal{C}(A|B, B|A) = A \dashv\vdash B$.

The assessment (x, y, z) on $\{A|B, B|A, A \dashv\vdash B\}$, with A, B logically independent, is coherent if and only if (Hamacher t-norm)

$$z = T_0^H(x, y) = \begin{cases} 0 & (x, y) = (0, 0), \\ \frac{xy}{x+y-xy} & (x, y) \neq (0, 0). \end{cases}$$

A simple proof when $P(A) > 0$, $P(B) > 0$*:*

$$z = P(AB \,|\, A \vee B) = \frac{P(AB)}{P(A)+P(B)-P(AB)} = \frac{\frac{P(AB)}{P(B)P(A)} \cdot P(AB)}{\frac{P(AB)}{P(B)P(A)} \cdot [P(A)+P(B)-P(AB)]} = \frac{xy}{x+y-xy}.$$

Other cases with logical dependencies:
AND, CM, CUT, OR rules.

(the conjunction, with the exception of OR rule, coincides with quasi conjunction)

(i) $H = K$

AND rule: $H \mid\!\sim A\,,\; H \mid\!\sim B \;\Rightarrow\; H \mid\!\sim AB$;

CM rule: $H \mid\!\sim A\,,\; H \mid\!\sim B \;\Rightarrow\; BH \mid\!\sim A$;

$(A|H) \wedge (B|H) = AB|H = \mathcal{C}(A|H, B|H) \subseteq A|BH\,;$

(ii) $H = BK$

CUT rule: $BK \mid\!\sim A\,,\ K \mid\!\sim B \ \Rightarrow\ K \mid\!\sim A$;

$$(A|BK) \wedge (B|K) = \mathcal{C}(A|BK, B|K) = AB|K \subseteq A|K\,;$$

(iii) $A = B$

Or rule: $H \mid\!\sim A\,,\ K \mid\!\sim A \ \Rightarrow\ H \vee K \mid\!\sim A$;

$$(A|H) \wedge (A|K) = (AHK + x \cdot AH^cK + y \cdot AHK^c) \,|\, (H \vee K)$$

$$\mathcal{C}(A|H, A|K) = (AHK + AH^cK + AHK^c) \,|\, (H \vee K) =$$

$$= A|(H \vee K) \ \geq \ (A|H) \wedge (A|K)\,;$$

Generalization and degradation of inference rules

A generalization of OR, CM and CUT rules, with an analysis of their degradation, has been given in (Gilio 2011).

For instance, concerning OR rule, it can be proved that the extension

$$z_n = P[A \,|\, (H_1 \vee \cdots \vee H_n)]$$

of the assessment

$$P(A|H_1) = x_1\,,\ \cdots\,,\ P(A|H_n) = x_n\,,$$

with $A, H_1, \ldots, H_n$ logically independent, is coherent if and only if $z'_n \leq z_n \leq z''_n$, with

$$z'_n \;=\; \frac{1}{1+\sum_{i=1}^{n}\frac{1-x_i}{x_i}}\,, \quad z''_n \;=\; \frac{\sum_{i=1}^{n}\frac{x_i}{1-x_i}}{1+\sum_{i=1}^{n}\frac{x_i}{1-x_i}}\,.$$

These formulas illustrate the degradation of OR rule when the number of premises increases.

For instance, if $x_1 = \cdots = x_n = \frac{1}{2}$, then it holds

$$[z'_2, z''_2] = \left[\frac{1}{3}, \frac{2}{3}\right]\,,\ \cdots\,,\ [z'_n, z''_n] = \left[\frac{1}{n+1}, \frac{n}{n+1}\right]\,.$$

If $x_1 = \cdots = x_n = x$, then

$$z'_n = \frac{x}{1+(n-1)(1-x)}\,, \quad z''_n = \frac{nx}{1+(n-1)x}\,,$$

$$z''_n - z'_n = \frac{(n^2-1)x(1-x)}{n+(n-1)^2 x(1-x)} \simeq \frac{x(1-x)}{\frac{1}{n} + x(1-x)}\,.$$

As we can see, given $x \in (0,1)$, if n is 'high', then: $z'_n \simeq 0\,,\ z''_n \simeq 1\,,\ z''_n - z'_n \simeq 1$; that is, inferences become very imprecise.

An apparent paradox on $(A|H) \wedge (B|K)$

We first observe that:
$(B|A) \wedge A = (B|A) \wedge (A|\Omega) = AB|\Omega = AB$;
then: $\mathbb{P}[(B|A) \wedge A] = P(AB) = P(B|A)P(A)$, and *it seems* that $B|A$ and A are stochastically independent.

Moreover, it also seems that, if $HK = \emptyset$, then $A|H$ and $B|K$ are *stochastically independent* (as observed in a discussion by D. Edgington).

This appears unreasonable; what can we say?

Actually, given the assessment

$P(A|H) = x\,,\ P(B|K) = y\,,\ \mathbb{P}[(A|H) \wedge (B|K)] = z\,,$

with $HK = \emptyset$, it can be proved that: $z = xy$; that is:

$$\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)P(B|K)\,. \qquad (2)$$

Proof:

As $HK = \emptyset$, it holds: $H^cK = K$, $HK^c = H$; then

$$(A|H) \wedge (B|K) = (x\,H^cBK + y\,AHK^c)\,|\,(H \vee K) =$$

$$= x\,BK\,|\,(H \vee K) + y\,AH\,|\,(H \vee K)\,;$$

then: $z = \mathbb{P}[(A|H) \wedge (B|K)] =$

$$= x \cdot P[BK \,|\, (H \vee K)] + y \cdot P[AH \,|\, (H \vee K)]\,,$$

and, as $P[H \,|\, (H \vee K)] + P[K \,|\, (H \vee K)] = 1$, by the compound probability theorem, we obtain

$$z = xy \cdot P[K \,|\, (H \vee K)] + xy \cdot P[H \,|\, (H \vee K)] = xy\,.$$

Does this equality mean that $A|H$ and $B|K$ are stochastically independent ?

We observe that:

• $(A|H) \wedge (B|K)$ is a *conditional random quantity*, not a conditional event; then the correct framework for giving a meaning to equality (2) is that of random quantities;

• $(A|H) \wedge (B|K) = x\, H^c BK + y\, AHK^c + z\, H^c K^c =$
$= x\, H^c BK + y\, AHK^c + xy\, H^c K^c = (A|H) \cdot (B|K)$;
(the conjunction coincides with the *product* of the *conditional random quantities* $A|H, B|K$);

• then, the equality (2) means *uncorrelation*, and not independence, between $A|H$ and $B|K$.

An example.

Given a random quantity $X \in \{1,2,3,4,5,6\}$, with

$$P(X=1)=p_1,\,\ldots,\,P(X=6)=p_6\,,$$

we set

$$A=(X\in\{2,4\})\,,\;\; H=(X\in\{1,2\})\,,$$

$$B=(X\in\{4,6\})\,,\;\; K=(X\in\{5,6\})\,.$$

We have: $HK=\emptyset$, with $P(H\vee K)=$

$$=P(X\in\{1,2,5,6\})=p_1+p_2+p_5+p_6\,,$$

and with

$$P(A|H)=x=\frac{p_2}{p_1+p_2}\,,\;\; P(B|K)=y=\frac{p_6}{p_5+p_6}\,.$$

Moreover

$$(A|H)\wedge(B|K)=x\,BK\,|\,(H\vee K)+y\,AH\,|\,(H\vee K),$$

with

$$BK = (X = 6)\,, \quad AH = (X = 2)\,,$$

$$P[BK \,|\, (H \vee K)] = \frac{p_6}{p_1 + p_2 + p_5 + p_6}\,,$$

$$P[AH \,|\, (H \vee K)] = \frac{p_2}{p_1 + p_2 + p_5 + p_6}\,.$$

Then

$$\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(A|H) \cdot (B|K)] = z =$$

$$= xP[BK \,|\, (H \vee K)] + yP[AH \,|\, (H \vee K)] = ,$$

$$= \frac{p_6 x}{p_1 + p_2 + p_5 + p_6} + \frac{p_2 y}{p_1 + p_2 + p_5 + p_6} =$$

$$= \frac{p_2 p_6}{p_1 + p_2 + p_5 + p_6}\left[\frac{1}{p_1 + p_2} + \frac{1}{p_5 + p_6}\right] =$$

$$= \frac{p_2}{p_1 + p_2} \cdot \frac{p_6}{p_5 + p_6} = xy = P(A|H)P(B|K)\,.$$