Games and Dependence in Logic

Jouko Väänänen

Helsinki and Amsterdam

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Outline

TRUTH

Semantic Game

$\mathcal{A} \models \phi$?

Semantic game of first order logic

- Players hold a formula, one player at a time.
- Each thinks that if he or she holds the formula, it is true. When the formula is atomic, the one who holds it wins if the formula is true, otherwise the opponent wins.
- At disjunction and existential quantifier the player who holds the formula chooses.
- At conjunction and universal quantifier the other player chooses.
- At negation the formula changes hands.

TRUTH

Semantic Game

$\mathcal{A} \models \phi$?

CONSISTENCY

Model Existence Game

$\exists\mathcal{A}(\mathcal{A} \models \phi)$?

SEPARATION

Ehrenfeucht–Fraïssé Game

$\exists\phi(\mathcal{A} \models \phi \text{ and } \mathcal{B} \not\models \phi)$?

TRUTH

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Beth tableaux

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Thin rigid models

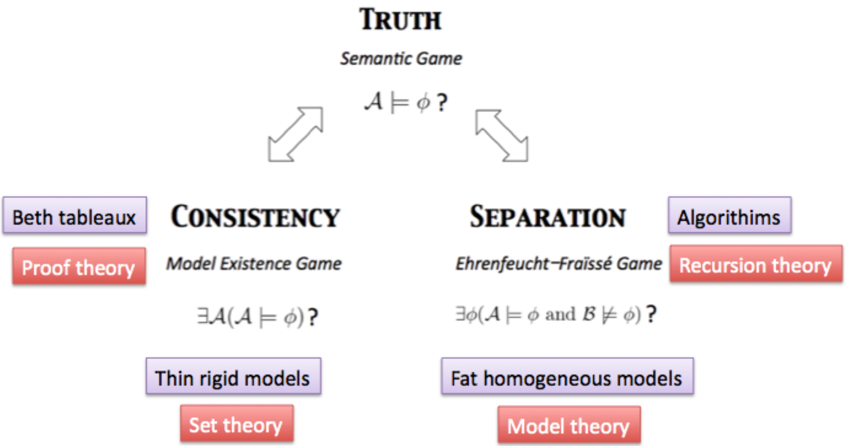
SEPARATION

Algorithims

Ehrenfeucht–Fraïssé Game

$\exists \phi(\mathcal{A} \models \phi \text{ and } \mathcal{B} \not\models \phi)$?

Fat homogeneous models


TRUTH
Semantic Game
$\mathcal{A} \models \phi$?
Beth tableaux
Proof theory
CONSISTENCY
Model Existence Game
$\exists \mathcal{A}(\mathcal{A} \models \phi)$?
Thin rigid models
Set theory
SEPARATION
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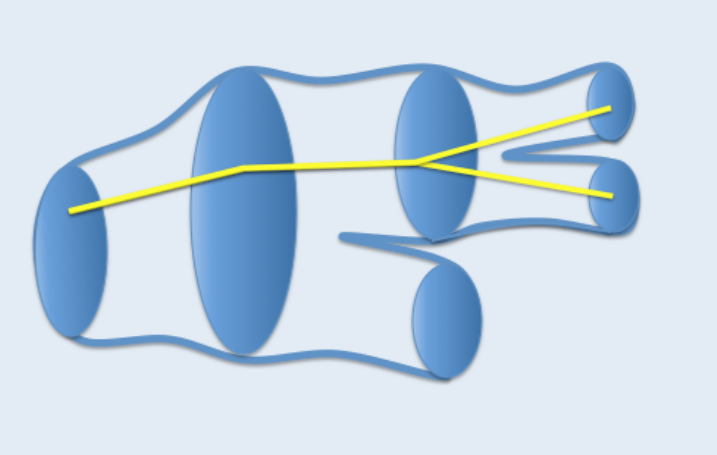
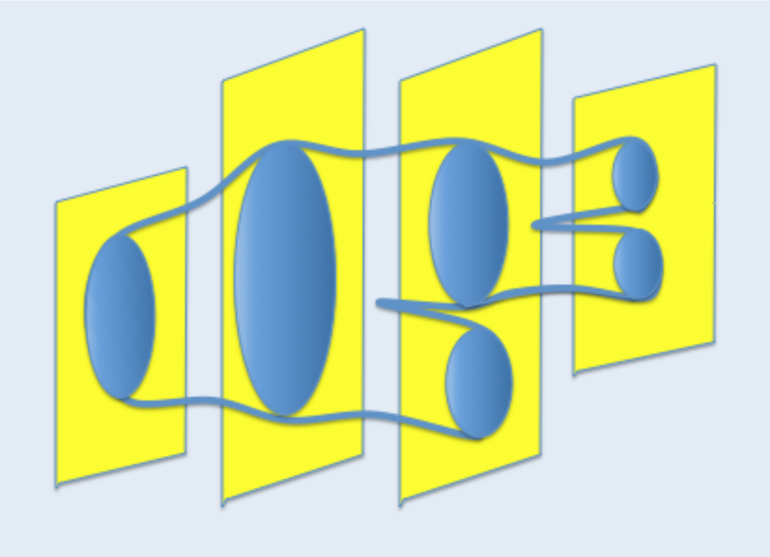
Ehrenfeucht–Fraïssé Game
Recursion theory
$\exists \phi(\mathcal{A} \models \phi \text{ and } \mathcal{B} \not\models \phi)$?
Fat homogeneous models
Model theory

Outline





- Strategy is a dependence relation $=(x_1, ..., x_n, y)$.
- Imperfect information game: $=(x_1, x_3, y)$.
- We may consider $=(x_1, ..., x_n, y)$ an atom.
- A player who holds $=(x_1, ..., x_n, y)$ wins, but a winning strategy has to obey the dependence.

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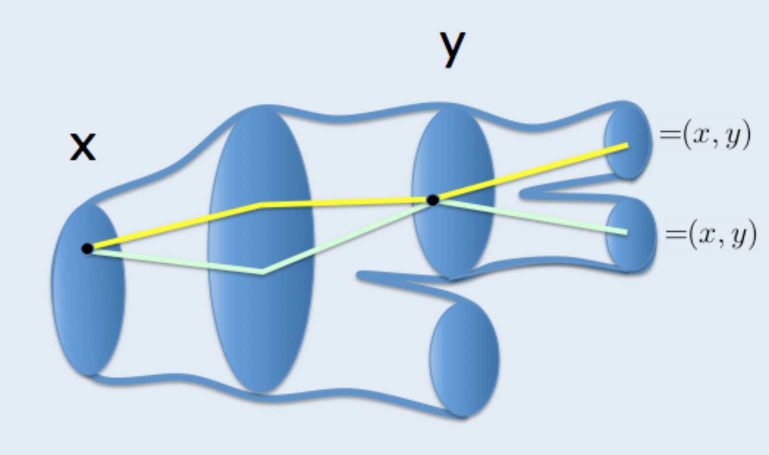
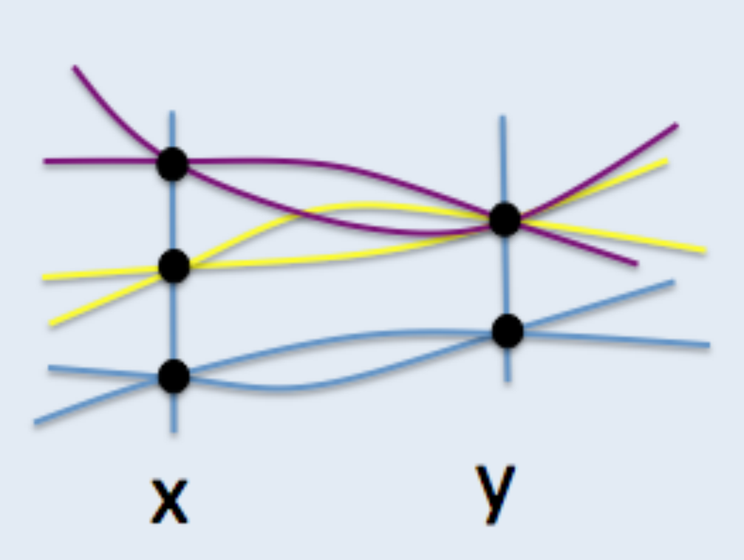
- Winning strategy has to obey the rule: If s and s' are two plays ending in $=\!(x_1, ..., x_n, y)$ in which a player has followed the strategy and s and s' give $x_1, ..., x_n$ the same value, they give also y the same value.


y
x
$=(x, y)$
$=(x, y)$


x
y

- The same for the independence atom $\vec{x} \perp_{\vec{z}} \vec{y}$.
- For a player to have a strategy which wins by holding an **independence** atom $\vec{x} \perp_{\vec{z}} \vec{y}$ the following has to be the case: If s and s' are two plays in which he or she followed the strategy and s and s' give $\vec{z}$ the same value, there is a third play s'' such that s'':
  - Follows the strategy,
  - Gives $\vec{z}$ the same value as s and s',
  - Gives $\vec{x}$ the same value that s gives,
  - Gives $\vec{y}$ the same value that s' does.

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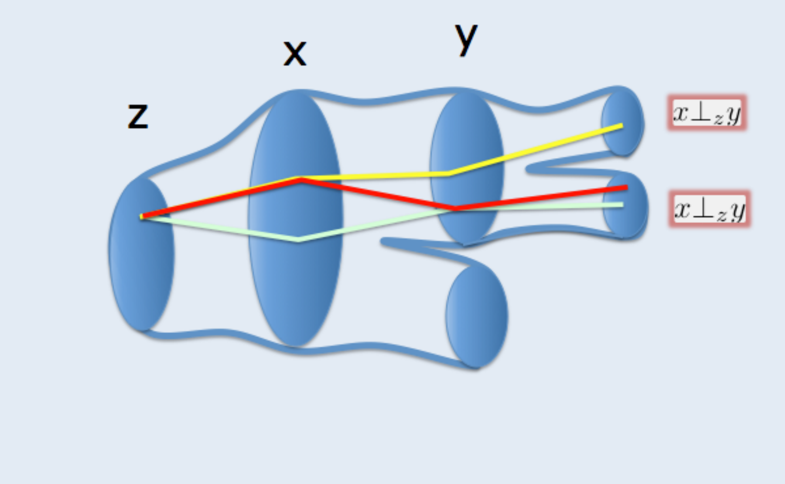
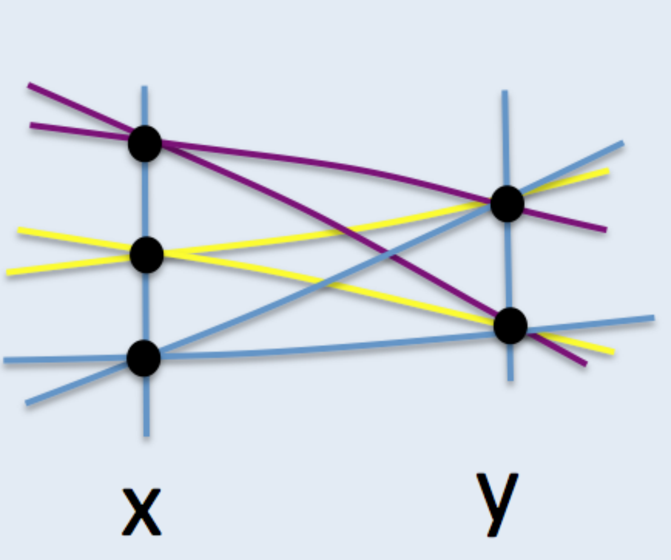

z
x
y
$x \perp_z y$
$x \perp_z y$


x
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What does it mean game theoretically?

- Dependence and independence logics with team semantics transcend the framework of game theoretic semantics.
- Dependence atoms occur in game theory in the form of imperfect information games.
- Do independence atoms occur in game theory?

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Outline

Social choice

- Individual voters are the variables.
- The values of these variables are the preference relations of the individuals x_i.
- An assignment = a profile.
- The social choice function is just another variable y.
- Independence from irrelevant alternatives: $=(P_{a,b}(x_1), ..., P_{a,b}(x_n), P_{a,b}(y))$.
- x_i is a dictator: $=(x_i, y)$.
- Arrow's axioms invoke dependence only, but the proof of Arrow's theorem depends on assumptions that invoke independence-type assumptions about the behavior of the electorate.

Social choice

- The proof of Arrow's theorem assumes—seemingly—that the social welfare function is defined for all profiles.
- Enough: the domain of the social welfare function manifests independence of the voters from each other in the sense that profiles where voters change their preferences (as needed in the proof) are also possible.
- Arrow's Paradox: Unlimited freedom leads to the situation that dictatorship is the only way to satisfy Arrow's axioms.

Outline

1. Three games of logic
   - Three games
   - Dependence and independence

2. Applications of team semantics
   - Social choice
   - Physics

Physics

- Samson Abramsky: "Relational Hidden Variables and Non-Locality".
- Team = a set of observations. Experiments $q_1, ..., q_n$. Each has an input and an output. Input of experiment q_i denoted x_i, and the output y_i. After m rounds of making the experiments q_1,...,q_m we have the data

$$X = \begin{array}{|ccccc|}
x_1 & y_1 & \dots & x_n & y_n \\
\hline
a_1^1 & b_1^1 & \dots & a_n^1 & b_n^1 \\
a_1^2 & b_1^2 & \dots & a_n^2 & b_n^2 \\
\vdots & \vdots & \dots & \vdots & \vdots \\
a_1^m & b_1^m & \dots & a_n^m & b_n^m
\end{array}$$

Determinism

- Team X supports **strong determinism** if it satisfies

$$=\!(x_i, y_i)$$

for all $i = 1, ..., n$.

- Team X supports **weak determinism** if it satisfies

$$=\!(x_1, ..., x_n, y_i)$$

for all $i = 1, ..., n$.

Empirical models

A dependence (and independence) logic formula $\phi(\vec{x}, \vec{y})$ describes a “relational model".

Example

$$\exists X \exists Y \exists a \exists b (x_1 = X \wedge x_2 = Y \wedge (y_1 = a \vee y_1 = b) \wedge (y_2 = a \vee y_2 = b))$$

says that in this model there are two experiments, both have a fixed input (X and Y), with two possible outcomes (a or b) for either experiment. An example of such a team would be

$$X = \begin{array}{|cccc|} x_1 & x_2 & y_1 & y_2 \\ \hline X & a & Y & b \\ X & b & Y & a \end{array}$$

Hidden variables

A **hidden variable model** is of the form

$$Y = \begin{array}{|cccccc|} x_1 & y_1 & \ldots & x_n & y_n & z \\ \hline a_1^1 & b_1^1 & \ldots & a_n^1 & b_n^1 & \gamma^1 \\ a_1^2 & b_1^2 & \ldots & a_n^2 & b_n^2 & \gamma^2 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ a_1^m & b_1^m & \ldots & a_n^m & b_n^m & \gamma^m \end{array}$$

where the γ^i are values of a hidden variable z.
A hidden variable model Y **satisfies** the relational model X (i.e. X is **realized** by Y) if

$$X = \exists z Y,$$

that is

$$s \in X \iff \exists s' \in Y(s'(x_1) = s(x_1) \wedge s'(y_1) = s(y_1) \wedge \ldots$$
$$s'(x_n) = s(x_n) \wedge s'(y_n) = s(y_n)).$$

Single-valuedness

A team X is said to support **single-valuedness of the hidden variable** z if z has only one value in the team.
We can express this with the formula

$$=(z).$$

Outcome-independence

An empirical team X is said to support **outcome - independence** if the following holds: Suppose the team X has two measurement-outcome combinations s and s' with the same total input data $\vec{x}$ and the same hidden variable z, i.e. $s(\vec{x}) = s'(\vec{x})$ and $s(z) = s'(z)$. We demand that output $s(y_i)$ should occur as an output also if the outputs $s(\{y_j : j \neq i\})$ are changed to $s'(\{y_j : j \neq i\})$.
We can express output-independence with the formula

$$y_i \perp_{\vec{x},z} \{y_j : j \neq i\}.$$

Other

- No-signaling.
- Independence of the hidden variable
- Parameter independence
- Locality

No-Go Results

- Einstein-Podolsky-Rosen result: There is an empirical model (team) which cannot be realized by any hidden variable models satisfying single-valuedness and outcome-independence.
- Other: Greenberger-Horne-Zeilinger result, Hardy paradox, Kochen-Specker Theorem.

Punchlines

- Game(s) permeate logic.
- Team semantics transcends games.
- The emergent logic of dependence and independence concepts provides a common mathematical and conceptual basis for phenomena such as Arrow's Paradox and the no-go-results of Quantum Mechanics.