SCIENTIFIC REPORT

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In my visit to the Kurt Gödel Research Centre in Vienna I worked with Dr. Misha Gavrilovich for two weeks. In the past Misha, alongside Assaf Hasson developed an idea of how to generate a model category structure from a model of ZFC. The goal of my visit was to study this method, purpose various improvements and see if we can express some large cardinal notions using this method.

During this time we also came up with the idea to apply a similar construction to classes closely related to PCF theory, in order to give algebro-geometric way to express PCF theory concepts as homotopy invariants of the associated homotopy category.

1. The Construction

Suppose that M is a set model of ZFC. For simplicity one can assume that $M = V_\kappa$ for an inaccessible κ, although the following construction does not even require the model to be well-founded.

We begin with a category called $St_\kappa(M)$. The objects of our category are classes of sets in M. An object, if so, is an element of $\mathcal{P}(\mathcal{P}(M))$, where $\mathcal{P}$ denotes the power set. There is a very delicate point which we will discuss later, whether we take externally definable classes ($\mathcal{P}(\mathcal{P}(M))$) or do we limit ourselves to internally definable collections, which is to say that we only take a very small portion of this collection.

In Vienna we began by taking the externally definable collection, but it turned out to be a mistake. It turns out that in order to express large cardinal related notions to elementary embeddings, it is best to work with the internally definable classes.

For two objects X and Y we have an arrow $X \to Y$ if and only if every element of X is a subset of some element of Y. The arrows, if so, are unique when they exist.

So far we have yet to mention the mysterious κ which appeared in the name of our category. We require that κ is a regular cardinal in M, and we give labels to the arrows as follows:

(c): $X \xrightarrow{(c)} Y$ if for every $y \in Y$ there exists $x \in X$ such that $|y| \leq |x \cap y| + \kappa$.

(wc): $X \xrightarrow{(wc)} Y$ if for every $y \in Y$ there exists $x \in X$ such that $|x \setminus y| < \kappa$.

We can now define (f) and (wf) arrows by lifting properties against (c) and (wc) arrows. And (w) arrows, of course, are both (wc) and (wf). This naive construction, unfortunately does not yield a model category. In order to have a model category we need to restrict ourselves to objects X for which the diagram below holds:
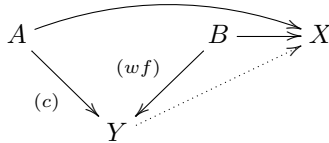
Where the fully drawn arrows are assumed, and the dotted arrow is derived.

The collection of these objects is the model-category denoted by $Qt_\kappa(M)$. Lastly we define $Ht_\kappa(M)$ as the localization of Qt_κ by (w) arrows.

2. External vs. Internal Constructions

When we began working, we noted that there could be a difference between the internal and external approach to this construction. Foolishly, or rather naively, we chose the external approach, because that what a naive person would do when given a set model. He would take everything he can.

Suppose that M is a model of ZFC and $N \subseteq M$ is an inner model of M. Let κ be a regular cardinal in both M and N we can construct two categories for N:

(1) We may construct $St_\kappa(N)$ and $Qt_\kappa(N)$ as above, disregarding the fact that N is an inner model of M; or

(2) we may construct the relative category whose objects are of the form $X \cap N$, when X is in $St_\kappa(M)$ (or $Qt_\kappa(M)$), and let the labels be inherited from the category of M. Let us denote the relative category as $St_\kappa(M) \cap N$ (despite being an abuse of notation, the meaning is clear).

It turns out that for the external construction in some cases where one would expect a difference between $St_\kappa(N)$ and $St_\kappa(M) \cap N$ to occur, there was in fact no difference between the categories. This was a disappointing discovery, albeit important. Alas due to our decision to work with the external construction for most of our time, we could not make a substantial progress in understanding the internal construction.

The internal construction itself is much more complicated to handle. This is because classes are defined by formulas and parameters. When passing to an inner model one can reinterpret the formula, and the result could be vastly different. For example let $\varphi(x)$ be a formula which states $V = L \wedge x = x$. If M is not a model of the axiom $V = L$ the class defined by this formula is empty, whereas reinterpreting this formula in L^M yields the entire universe.

Furthermore, if we do not allow parameters in the formulas then we only have countably many classes which are definable. This makes quite a boring life, so to spice things up one permits parameters. However we cannot reinterpret a formula with parameters from $M \setminus N$ in N.

Working with internally definable classes requires a delicate care for the fine details, if so, and we have yet to find a very natural way of expressing the interpretation of a formula (with parameters) in the language of categories and diagrams.

Not all is bad when working with internally definable classes. For example, we conjecture that if $j \colon M \to N$ is an elementary embedding of M into an inner model N, then j induces a Quillen adjunction between $Qt_\kappa(M)$ and $Qt_{j(\kappa)}(N)$.

In fact, we further conjecture that every Quillen adjunction $F \colon Qt_\kappa(M) \to Qt_\kappa(N)$ respecting reinterpretation is induced by an elementary embedding. To fully formalize this conjecture a good notion of interpretation as a functor should be established.

3. PCF Theory

In PCF theory one begins with a set A whose members are regular cardinals, and $|A| < \min A$. We ask, given an ideal I on A, what is the cofinality of $\prod A/I$ if it exists? What are the possible cofinalities when we vary over the proper ideals on A.

To represent PCF theory concepts we first need a suitable framework. One can look at the St_κ construction and consider it as a construction coming from a lattice $(P, \leq)$ where the category $St(P)$ is such that its objects are subsets of P (excluding the empty set if it bothers us), and the arrows are defined uniquely in the following manner:

$$X \to Y \iff \forall x \in X \exists y \in Y : x \leq y.$$

One can now identify the κ subscript as the ideal of sets of size of at most κ, and the labels in $St_\kappa(M)$ represent in some sense how the arrows would be calculated if we would take inclusion modulo that ideal. Similarly in our general case, if I is an ideal over P (in the lattice-theoretical sense of the word), then we can ask whether or not two arrows would be identified modulo I. So in its general form we say that in $St_I(P)$ we define the following labels:

(c): $X \xrightarrow{(c)} Y$ if for every $y \in Y$ there exists $x \in X$ such that $y \leq_I x \wedge y$.

(wc): $X \xrightarrow{(wc)} Y$ if for every $y \in Y$ there exists $x \in X$ such that $y <_I x$.

We return to our PCF theoretical constructions, and we observe that Ord^A is a lattice, where $g \leq f$ if and only if $g(a) \leq f(a)$ for all $a \in A$. This makes a quasi-order, but the above definitions are good for this too. Furthermore we can define the meet and join operators as pointwise min and max operators. For simplicity we shall denote $St_I(A)$ the category constructed as above where I is the ideal over Ord^A induced by an ideal I' over A (and we will never again make this distinction between I and I').

Typical PCF question is whether or not a set $F \subseteq \mathrm{Ord}^A$ without a maximum has true cofinality modulo I, namely is F/I equivalent to a linearly ordered set? Using the above notation it means that there exists a (wc) arrow between F and some linearly ordered F' in the category $St_I(A)$.

Furthermore, if I is an ideal over A we can identify the kernel of the functor from $St(A)$ to $St_I(A)$. It is the class of all functions which are zero outside I. We can now express basic PCF theorems as statements of universal properties, and chain complexes passing from $St_I(A)$ to $St_J(A)$ for ideals $I \subseteq J$.

For example, consider the bounding number of I, defined as

$$\mathfrak{b}(I) = \min\{\kappa \mid \exists B \subseteq \prod A, B \text{ unbounded} \mod I, |B| = \kappa\},$$

namely the smallest size of a subset of the product of all cardinals in A which is unbounded modulo the ideal I.

PCF theory tells us that if there is an ideal I such that $\mathfrak{b}(I) = \kappa$ then there is a minimal ideal I_κ with this property, namely $\mathfrak{b}(I) = \kappa \implies I_\kappa \subseteq I$. It also tells us that $\{\mathfrak{b}(I) \mid I \text{ is a proper ideal over } A\}$ is an interval (sans singular cardinals), and the difference between the minimal ideal for κ and for κ^+ is a particular subset of A which can be characterized.

We can therefore express the minimality as a universality property. Passing from one minimal ideal to another, if so, is characterized by a single set added to the ideal. Actually this set can be characterized by a single function from A to the ordinals too. We hope that the sequence of PCF ideals in some sense analogous to the sequence of cohomology groups of a chain complex, and the construction above is analogous to its (co)boundary operator.

4. Future Work

The future work should be dedicated to developing the following ideas:

(1) Investigate a better expressibility of the internal definitions via a functor from the category of formulas to the universe (i.e. the interpretation functor). Understanding this functor is crucial for expressing elementary embeddings (which commute with it in an obvious way).

(2) Continue to explore the set theoretical connection between $Qt_\kappa(M)$ and $Qt_\kappa(N)$ when N is an inner model of M. In particular the relation between the assertion that the covering lemma holds for N, and naturally occurring functors between the two categories.

(3) Two models of ZFC with the same ordinals are equal if and only if they have the same sets of ordinals. In this aspects it might sufficient, and perhaps more convenient, to discuss constructions as St_κ limited to classes of sets of ordinals.

(4) Recent work (by Laver and Hamkins) has shown that the ground model is definable with parameters in generic extensions. In particular it means that if we construct $St_\kappa(V[G])$, where $V[G]$ is a generic extension of V, then V is a definable class and therefore an object of this category. It is possible that there is a strong relation between the properties of the forcing (e.g. preservation of cofinalities) and the corresponding categories.

(5) To continue and develop the PCF theory expressibility via these categories. Some tools are still missing for this task, e.g. the expressibility of linearly ordered family of functions. We would like to develop these tools and continue to explore and formulate PCF statements in this settings.

Naturally our work for the time being will take a slower pace, due to the fact that in the apparent future it is going to be limited to email correspondence and be subjected to a busier schedule including other academic activities.

I hope that solving the first problem will allow for an exact formulation of other conjectures and problems in a way which suited for proper discussion with the rest of the mathematical community.

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