Report for *INFTY – Short Visit Grant 4935*

Project Title: Computational complexity in set theory

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Purpose of visit

The purpose of this visit was to collaborate on topics which combine complexity theory and computation theory with set theory. Specifically, our aim was to develop set-theoretic analogs of the time and space complexity classes of computational complexity using schematic, proof-theoretic and machine-theoretic definitions of these classes.

Initial progress had already been achieved in [3], which introduced the class of safe recursive set functions (SRSF) by an adaptation of the scheme of Bellantoni-Cook style safe-recursion for polynomial time computation. For suitable encodings of hereditarily finite sets, the SRSF functions were shown to be the functions computable by alternating exponential time Turing machines with polynomially many alternations. For arbitrary sets, the SRSF functions were characterized with Jensen's J-hierarchy, and for binary omega-strings shown to be equivalent to functions computed by infinite time Turing machines [4] in time less than ω^ω.

Description of the work carried out during the visit

In preparation of the visit, we reviewed relevant literature, in particular [1, 5, 6, 2]. During the visit, we started by discussing whether it is possible to define a class of functions on sets which is close to the class of *predicatively computable set functions (PCSF)* as defined by Arai [1], and thus resembles polynomial time computations on hereditarily finite sets. We were looking for a class which is not based on safe-recursion as PCSF and our SRSF [3], but is in Cobham-style based on limited recursion. For this to work, we first needed a suitable definition of *limiting* functions on sets, which in the arithmetic setting are given for example by the so called *smash* function. We proposed several such smash functions on sets, and analyzed their properties.

The second main topic of our collaboration was to find possible weak set theories, which capture complexity classes through their provably recursive functions. Weak arithmetic theories with such properties are well known, they are given by sub-theories of Bounded Arithmetic, and were first introduced by Sam Buss. For set theories, Arai [1] recently proposed weak set theories which are linked to his PCSF functions, and, therefore, have built-in means to distinguish between normal and safe values, a feature which is necessary to define safe-recursion. We investigated ways of having such weak set theories without making use of normal and safe values. The starting point of our work was Rathjen's characterization of the primitive recursive set functions as the provable recursive functions

in a restricted fragment of Kripke Platek (KP) set theory [5], together with the Cobham-style definition of polynomial time computable set functions which we obtained in the first part of our visit.

Description of the main results obtained

We have achieved the following main results: We have defined several candidate smash functions on sets, or more precisely on graph representations of sets. They all have the property that cardinalities and ranks are only increased polynomially. They differ in that the structure of the created graphs varies in complexity. Currently it is unclear whether each of them is sufficient to define a class of functions under Cobham's limited recursion scheme which captures Arai's PCSF functions. However, we are convinced that this is true for the strongest candidate smash function (in the above sense.) We have denoted our class of set functions the *Cobham recursive set functions (CRSF).*

Based on our smash functions, we have considered several restrictions of Rathjen's fragment of KP for the primitive recursive set functions. We have established several witnessing arguments to show that the provably recursive functions of our weak set theory correspond to our CRSF functions. On the way to establishing this connection, we discovered that a form of choice is needed if the theories contain standard forms of foundation. We have established a forcing argument that eliminates global choice functions in the presence of the Axiom of Choice. It remains to be seen whether choice can be completely eliminated in the presence of these standard forms of foundation.

Future collaboration with host institution

Our plan is to continue our successful collaboration, in order to obtain a full picture on how to resemble computational complexity hierarchies within set theory.

Projected publications

We have planned two publications which shall come out of this research collaboratoin. The first will deal with the first question which we worked on, discussing various forms of smash functions on sets, and using these to give a Cobham style characterization of Arai's PCSF functions via limited recursion. The second paper will deal with weak subtheories of Rathjen's fragment of KP, and use witnessing arguments to characterize their provably recursive functions as the Cobham recursive set functions defined in the first paper. Furthermore, it will discuss the role of choice in this endavour.

References

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