Technical report for INFTY exchange visit no. 4144
(Talagrand's non-measurable Maharam algebra)
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Aim of visit. The aim of this visit was to begin a cooperation with B. Balcar and J. Starý to investigate Talagrand's construction of a non-measurable Maharam algebra from [Tal08] and related topics.

Main. B. Balcar, J. Starý and the author began a review of the current literature concerning Maharam submeasures and Maharam algebras. The papers under consideration were [BH01], [BGJ98], [BFH99], [BJP05], [BJ08], [BP10] and also T. Pazák's doctoral thesis [Paz06]. In [BJ08] and [Frea] it was proved that if $\mathfrak{A}$ is a measure algebra and $\mathfrak{B}$ is an $\mathfrak{A}$-name for a non-measurable Maharam algebra, then the two step iteration $\mathfrak{A} * \mathfrak{B}$ is also a non-measurable Maharam algebra. This is an interesting result since it shows that there are class many non-measurable Maharam algebras (for each cardinal κ take $\mathfrak{A}$ to be a measure algebra of cardinality κ and $\mathfrak{B}$ to be Talagrand's example). Together with B. Balcar and J. Starý the author proved the following converse result.

Lemma 1. *Let $\mathfrak{A}$ be a complete Boolean algebra and $\mathfrak{B}$ be an $\mathfrak{A}$-name such that*

$$||\mathfrak{B} \text{ is a complete Boolean algebra}|| = 1.$$

If the iteration $\mathfrak{A} * \mathfrak{B}$ is a Maharam algebra then $||\mathfrak{B}$ is a Maharam algebra$|| = 1$.

The proof of this lemma uses Todorčević's characterisation from [Tod04] that a ccc complete Boolean algebra is a Maharam algebra if and only if it is weakly distributive and σ-finite-cc.

This is a relevant result because it is a basic forcing fact that if $\mathfrak{C}$ is a complete Boolean algebra that contains $\mathfrak{A}$ as a complete subalgebra then there exists an $\mathfrak{A}$-name $\mathfrak{B}$ such that $\mathfrak{C}$ is the iteration $\mathfrak{A} * \mathfrak{B}$ (see [Jec03, Page 269]). Recall that it is still an open question if there exists a Maharam algebra that does not contain a measure algebra as complete subalgebra (see [FV07, Question 12], [Freb, Problem 3A], [Vel09, Question 3]). Lemma 1 may be useful in such an investigation since if any given Maharam algebra $\mathfrak{C}$ does contain a measure algebra $\mathfrak{A}$ as a complete subalgebra then perhaps the Maharam algebra $\mathfrak{B}$, obtained in the above way, can witness a new example. Unfortunately, it seems that this lemma is folklore, although the author could not find a reference for it.

During this visit the author also learnt of a theorem of Choquet which, given a locally compact second countable Hausdorff space X, characterises those functionals on the collection of compact sets $\mathbb{K}$ of X that can be realised as a probability measure defined on a σ-subalgebra of $\mathcal{P}(\mathbb{K})$ (see [Mat75, Theorem 2.2.1]). A countable version of this result was obtained by the author in [Sel12] and the author was able to strengthen this result to arbitrary cardinality. Specifically the author was able to prove the following, which may be viewed as an abstract (and stronger) form of Choquet's theorem.

Theorem 2. *For any Boolean algebra $\mathfrak{A}$ there exists a Boolean algebra $\mathfrak{B}$ and an injective union preserving map $f : \mathfrak{A} \to \mathfrak{B}$ such that for any functional $T : \mathfrak{A} \to \mathbb{R}$ there exists a signed measure $\lambda : \mathfrak{B} \to \mathbb{R}$ such that $T = \lambda \circ f$.*

We remark that Theorem 2 was already obtained by S. Solecki in [Sol00] using techniques from [Mat75]. The author's proof uses techniques from [Sel12] which are different. Let us now briefly describe the proof of Theorem 2. In [Sel12] the following lemma was proved.

Lemma 3. *For each finite Boolean algebra $\mathfrak{A}$, let $\mathrm{Fr}^*\mathfrak{A}$ be the Boolean algebra $\mathcal{P}(\mathcal{P}^+(\mathrm{atoms}(\mathfrak{A})))$ and consider the map $f : \mathrm{atoms}(\mathfrak{A}) \to \mathrm{Fr}^*\mathfrak{A}$ which sends each $a \in \mathrm{atoms}(\mathfrak{A})$ to the set $\{y \in \mathcal{P}^+(\mathrm{atoms}(\mathfrak{A})) : a \in y\}$. Extend f to $\mathfrak{A}$ by sending 0 to 0 and $\bigcup X$ to $\bigcup_{a \in X} f(a)$, for each $X \subseteq \mathrm{atoms}(\mathfrak{A})$. Call* *a map constructed in this way* $\mathfrak{A}$-**good**. *Then for any functional* $T : \mathfrak{A} \to \mathbb{R}$ *there exists a unique signed-measure* $\lambda : \mathrm{Fr}^*\mathfrak{A} \to \mathbb{R}$ *such that* $T = \lambda \circ f$.

Given two Boolean algebra $\mathfrak{A}$ and $\mathfrak{B}$ call a map $f : \mathfrak{A} \to \mathfrak{B}$ **good** if and only if $\mathfrak{B} = \langle f[\mathfrak{A}]\rangle$ ($\mathfrak{B}$ is generated by the image of $\mathfrak{A}$ under f) and for every finite subalgebra $\mathfrak{A}' \subseteq \mathfrak{A}$ there exists an isomorphism $g : \langle f[\mathfrak{A}']\rangle \to \mathrm{Fr}^*\mathfrak{A}'$ such that $g \circ f$ is $\mathfrak{A}'$-good.

Definition 4. ([Sol00], see also [Mat75, Lemma 2.2.1]) *If* $\mathfrak{A}$ *is a subalgebra of sets of a non-empty set* X *then let* $\mathcal{S}(\mathfrak{A})$ *be the subalgebra of* $\mathcal{P}([X]^{<\omega})$ *generated by sets of the form* $\{Y \in [X]^{<\omega} : Y \cap A \neq 0\}$, *for* $A \in \mathfrak{A}$.

Now given a Boolean algebra $\mathfrak{A}$ with Stone space X, the map $A \mapsto \{Y \in [X]^{<\omega} : Y \cap A \neq 0\} : \mathfrak{A} \to \mathcal{S}(X)$ is good. Theorem 2 now follows from the following extension of Lemma 3.

Theorem 5. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are Boolean algebras and* $f : \mathfrak{A} \to \mathfrak{B}$ *is good then for any functional* $T : \mathfrak{A} \to \mathfrak{B}$ *there exists a unique signed-measure* $\lambda : \mathfrak{B} \to \mathbb{R}$ *such that* $T = \lambda \circ f$.

The signed-measure λ is obtained as follows. Given $b \in \mathfrak{B}$ let $\mathfrak{A}'$ be a finite subalgebra of $\mathfrak{A}$ such that $b \in \langle f[\mathfrak{A}']\rangle$ and let $g : \langle f[\mathfrak{A}']\rangle \to \mathrm{Fr}^*\mathfrak{A}'$ be the promised isomorphism. Let $\lambda' : \mathrm{Fr}^*\mathfrak{A}' \to \mathbb{R}$ be the signed-measure promised by Lemma 3. Now let $\lambda(b) = \lambda'(g(b))$. One needs to check that this is well defined but this follows from the uniqueness claim in Lemma 3.

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