SELECTION PRINCIPLES AND DENSE SETS
SCIENTIFIC REPORT

1. Purpose of the visit and Introduction

For topological terminology we follow [9] while for set-theoretic terminology we follow [11].

From July 11 to July 21 I, Santi Spadaro of Ben Gurion University of the Negev in Be'er Sheva, Israel (Department of Mathematics) visited the Rényi Institute of the Hungarian Academy of Science in Budapest, Hungary and collaborated with Lajos Soukup and Dániel Soukup on *selection principles for dense sets*. With this term we indicate several properties roughly stating that a certain kind of *small* dense set can be obtained by diagonalizing over a countable sequence of dense sets. For example a space is said to be R-separable [3] if, given a countable sequence (D_n) of dense sets, one can obtain a dense set by picking a single point from each D_n. These properties admit elegant characterizations on spaces of continuous functions and hyperspaces (see, for example, [4], [3] and [15]) and allow for topological characterizations of well-known cardinal invariants of the continuum. For example, the least number of nowhere dense sets required to cover the real line is equal to the least weight of a countable non R-separable space (see [3] and [16]). R-separability also admits a natural game version where two players alternate picking dense sets and points and player II wins if the set of points picked is dense. Scheepers [16] proved that this game is essentially equivalent to the point-picking game of Berner and Juhász [7]. In particular, player II has a winning strategy in the R-separable game if and only if it has a *countable* π*-base* (that is, the POSET of all its non-empty open sets under containment has a countable dense set).

2. Description of the work carried out during the visit

We focused on two recent variants of this property: namely, we call a space D-separable (respectively, NWD-separable) if for every sequence (D_n) of dense sets we can find discrete (respectively, nowhere dense) sets $E_n \subset D_n$ such that $\bigcup_{n<\omega} E_n$ is dense. D-separability was first introduced and studied in [5]. In addition we call a space d-separable (nwd-separable) if it has a σ-discrete (σ-nowhere dense) dense subset. Clearly, every D-separable space is d-separable and every NWD-separable space is nwd-separable. A natural class of D-separable spaces is the class of spaces having a σ-disjoint π-base (which includes metric spaces). In studying these properties we have been in part inspired by past study on R-separability. Indeed, we have found connections with classical cardinal invariants of the continuum and exploited the affinity with topological games. But the achievement of this required much more than a straightforward generalization of the techniques available for R-separability, due to the completely different flavor of these two new properties. Our work can be broken up into three phases. Although considered in the order below, they were developed in a simultaneous way. The second phase gave rise to interesting side results on the so-called *discrete generability* (see, for example, [6] and [8]).

The first task we addressed was to distinguish between NWD-separability (*nwd*-separability) and D-separability (d-separability) and to construct good examples of non-NWD-separable spaces. These objectives are far from trivial, and while we were able to construct ZFC examples of NWD-separable non-D-separable spaces we still fail to have countable and compact examples. We do however have a compact *nwd*-separable non-d-separable space, as well as countable non-NWD separable spaces of small weight. We achieved these tasks by introducing new ad-hoc constructions of topological spaces, as well as revisiting known techniques like the *$\mathcal{D}$-forced spaces* of [12].

Then we investigated the influence of convergence on selective versions of separability, continuing the work of [2], [5] and [10]. One of the main convergence properties is the so-called *Fréchet property*, which states that every point in the closure of a set can be approximated by a convergent sequence. Building on results of Barman and Dow [2], Gruenhage and Sakai [10] were the first to note that every separable Fréchet space is R-separable. We have a consistent construction of a d-separable first-countable space which is not D-separable, showing that the Barman-Dow-Gruenhage-Sakai theorem cannot be extended to the context of D-separability. A weak convergence-type property which is natural to consider in our study is the so-called *discrete generability*: a space is called *discretely generated* if for every set A and a point $x \in \overline{A}$ there is a discrete set $D \subset A$ such that $x \in \overline{D}$. In the presence of some good separation, this property can make a d-separable space be D-separable (see [5]). Leandro Aurichi, Lucia Junqueira and Rodrigo Roque Dias in [1] were able to isolate a natural strengthening of discrete generability which implies D-separability of a d-separable space with no additional conditions. They call a space *discretely discretely generated* if for every discrete D which is contained in the closure of a set A, the set A contains another discrete E whose closure contains D. A byproduct of our study was the solution to some problems regarding discrete generability and related notions. As an intermediate step of some of our constructions, we were able to find the first consistent examples of non-discretely generated *P-spaces* (=G_δ sets are open) and of a discretely generated compact space without a point-countable π-base (thus partially solving problems from [6] and [8]). Moreover, we used the Aurichi-Junqueira-Roque Dias *discrete discrete generability* to find a partial answer to the following deceivingly innocent open problem, first posed by Mikhail Matveev: is $\sigma(2^{\aleph_1})$ D-separable? ($\sigma(2^{\aleph_1})$ is the subset of all functions of $2^{\aleph_1}$ having finite support). While we know that consistently, this space is D-separable, it is still open whether it can fail to be D-separable. However, we were able to establish the independence from ZFC of $\sigma(2^{\aleph_1})$ being discretely discretely generated. This has independent interest (pun unintended), especially since such a space is discretely generated in ZFC. However, it does not rule out the possibility that $\sigma(2^{\aleph_1})$ might be D-separable in ZFC. As a matter of fact, we even have R-separable spaces that fail to be generated by discrete sets.

Finally, we addressed the influence of metric-like properties on selective separability. Every metric space is D-separable (and hence also NWD-separable). One of the main line of investigation of both [1] and [5] is the extent to which the assumption about the *metric* can be weakened. One of the most important generalizations of metric spaces is the class of monotonically normal spaces. Its importance comes in part from the fact that it also generalizes the class of linearly ordered spaces and from Mary Ellen Rudin's deep structure theorem for compact linearly ordered spaces [14], which uses this notion.We improved results from [1] by showing that monotonically normal nwd-separable spaces are D-separable.

3. Description of the main results obtained

Here is a detailed list of our main results, which regard selective versions of separability, discrete generability and their interplay (see the above section for definitions and motivation).

(1) If the cofinality of the meager ideal is equal to $\aleph_1$ then the minimum cardinal κ such that 2^κ does not contain a countable NWD-separable dense subspace is $\aleph_1$.
(2) The least cardinal κ such that 2^κ is not D-separable is $\aleph_1$ in ZFC.
(3) There are ZFC examples of d-separable non-NWD-separable spaces.
(4) In every model obtained by adding $\aleph_2$ many Cohen reals to a ground model of GCH the space $\sigma(2^{\aleph_1})$ is D-separable.
(5) MA_{ω_1} implies that $\sigma(2^{\aleph_1})$ is discretely discretely generated.
(6) Under Jensen's Axiom $\diamondsuit$, $\sigma(2^{\aleph_1})$ fails to be discretely discretely generated.
(7) It is consistent that there are first-countable d-separable non-D-separable spaces (answers a question of Bella, Matveev and Spadaro from [5]).
(8) Every monotonically normal nwd-separable space is D-separable.
(9) There are compact nwd-separable non-d-separable spaces.
(10) There are consistent examples of compact discretely generated spaces without point-countable π-bases (answers a question of Dow, Tkachenko, Tkachuk and Wilson from [8]).
(11) There are R-separable spaces which are not discretely generated.
(12) It is consistent that the countably supported box-product of $\aleph_2$ many copies of the two-point discrete space is not discretely generated.

4. Future collaborations with the host institution

In certain cases D-separability or NWD-separability can be replaced by their game-theoretic strengthenings (that is, player II has a winning strategy in the respective game) in the above results. These are not as easy to characterize as in the case of R-separability. Based on analogy, one would be tempted to conjecture that a space is game-D-separable if and only if it has a σ-disjoint π-base. But a countable maximal space is a counterexample to that (see [5]). So the class of spaces where player II has a winning strategy in the D-separability game is interesting on its own and we plan to continue studying it in the future.

The following list of questions, which arose during my visit, will provide further material for future collaborations:

(1) Is there a compact D-separable space without a σ-disjoint π-base?
(2) Can $\sigma(2^{\aleph_1})$ consistently fail to be D-separable? For example, is this the case under $\diamondsuit$.
(3) Is there a ZFC example of a countable R-separable space of uncountable π-weight?
(4) (Juhász) Is there a ZFC example of a space where the D-separable game is undetermined?

(5) Are there countable NWD-separable non-D-separable spaces?
(6) Is there a compact separable discretely generated space without a countable π-base?
(7) Is there a ZFC example of a regular non-discretely generated P-*space*?

The first question appears in [5] and is inspired by the result that compact R-separable spaces have a countable π-base (see [4]). The third question is also unknown in the following stronger form: is there a ZFC example of a Fréchet countable space with uncountable π-weight? This version of our question was independently asked by Juhász to Gruenhage a few years ago and, since weight and π-weight are equal in topological groups, it looks similar to Malykhin's old problem of whether there is a ZFC example of a countable non-metrizable Fréchet topological group (see [13]).

5. Projected publications resulting from the grant

As a result of our collaboration, we gained a much better picture of these properties and we are writing up our results in a joint paper (D. Soukup, L. Soukup and S. Spadaro, *Selective versions of separability and discretely generated spaces* (working title)).

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