SCIENTIFIC REPORT FOR SHORT VISIT: *FORCING AXIOMS AND TOWER FORCING*

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1. Introduction

During this short visit (November 6 to November 13, 2011), we worked on problems related to forcing axioms, stationary set reflection, and tower forcing. The main focus was to answer some questions left open by our joint paper [2]. We made significant progress on these questions (see Section 2); of course, the paper resulting from this work will gratefully acknowledge support from the grant for the Short Visit.

A *tower of ideals of height* δ is a δ-sequence of ideals with certain coherence properties; the tower is called *presaturated* if (roughly) generic ultrapowers using the tower resemble almost-huge embeddings. Woodin proved that there are several natural towers of height δ which are definable over V_δ and are presaturated whenever δ is a Woodin cardinal. On the other hand, compactness properties of the critical point of the tower can sometimes conflict with presaturation of the tower:

1. Burke [1] showed that, if κ is supercompact and $\delta > \kappa$ is inaccessible, then there is a tower of height δ with critical point κ which is not even precipitous.
2. Foreman-Magidor [4] proved that strong forcing axioms like MM and PFA conflict with some nice ideals/towers with critical point ω_2 (e.g. the IA-stationary tower is not presaturated; there is no presaturated ideal on ω_2; $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ fails).

2. Stationary reflection and towers

In our joint paper [2], we proved more theorems along these lines. To state the theorems and questions, we need a few definitions.

Definition 1. *A set $M \prec H_\theta$ is called:*

- ω_1-guessing (G_{ω_1}) *iff (H_M, V) has the ω_1-approximation property as defined in Hamkins* [5]*, where H_M is the transitive collapse of M;*
- ω_1-internally club (IC_{ω_1}) *iff $M \cap [M]^\omega$ contains a club in $[M]^\omega$;*
- ω_1-internally stationary (IS_{ω_1}) *iff $M \cap [M]^\omega$ is stationary in $[M]^\omega$;*
- ω_1-internally unbounded (IU_{ω_1}) *iff $M \cap [M]^\omega$ is $\subset$-cofinal in $[M]^\omega$.*

GIC_{ω_1} denotes the class $G_{\omega_1} \cap IC_{\omega_1}$; similarly for GIS_{ω_1} and GIU_{ω_1}.

(Note: Viale and Weiss [11] proved that under PFA, $GIC_{\omega_1} \cap \wp_{\omega_2}(H_\theta)$ is stationary for all regular $\theta \geq \omega_2$; moreover many consequences of PFA factor through this result.)

In [2] we proved the following theorems, which are along the lines of the Burke and Foreman-Magidor results mentioned above.

Theorem 2. *(Cox-Viale [2]) The Reflection Principle at ω_2 ($RP([\omega_2]^\omega)$) implies there is no presaturated tower of ideals which concentrates on GIC_{ω_1}.*

Theorem 3. *(Cox-Viale [2]) The Strong Reflection Principle at ω_2 ($SRP([\omega_2]^\omega)$) plus the Tree Property at ω_2 ($TP(\omega_2)$) implies there is no presaturated tower of ideals which concentrates on GIS_{ω_1}.*

(Note this conclusion is stronger than the conclusion of Theorem 2 because $GIC_{\omega_1} \subseteq GIS_{\omega_1}$).

Combined with results from Viale-Weiss [11], these theorems imply that under PFA^+ or MM there are always non-presaturated towers with critical point ω_2; note the similarity of this result with the results of Foreman-Magidor mentioned above.

We had also shown the following, which shows that in one sense, the theorems above are sharp:

Theorem 4. *(Cox-Viale [2]) The word "presaturated" in Theorems 2 and 3 cannot be replaced by "precipitous". In particular, MM^{++} is consistent with a precipitous tower which concentrates on GIC_{ω_1}.*

In [2], we asked the following questions; essentially, asking if the results above were sharp (in ways different from the sharpness obtained in Theorem 4):

Question 5. *Is it consistent at all (i.e. with ZFC) to have a presaturated tower that concentrates on GIC_{ω_1}? Is such a tower consistent with PFA?*

Note that PFA^+ implies RP, so by Theorem 2 one cannot hope to obtain PFA^+ in Question 5.

Question 6. *Is it consistent with PFA^+, or even just with RP, to have a presaturated tower that concentrates on GIS_{ω_1}?*

Question 7. *Is Martin's Maximum (or just SRP plus $TP(\omega_2)$) consistent with a presaturated tower which concentrates on GIU_{ω_1}?*

During the Short Visit, we made significant progress on Questions 5, 6, and 7. The recent preprint of Neeman [7] appears very useful in answering these questions. We proved, using Neeman's so-called "sequence poset" (of models of 2 types) below an almost huge cardinal, that ZFC is consistent with a presaturated tower on GIC_{ω_1}; this partly answers Question 5. Moreover, though we do not know if such a tower is consistent with PFA, we did prove that it is consistent with a forcing axiom weaker than PFA (which, though formally weaker than PFA, has many of the same consequences as PFA which are widely believed to have consistency strength of a supercompact cardinal). We are currently making further modifications of Neeman's poset to provide affirmative answers to questions 6 and 7.

During the visit, we also noticed that the hypothesis of Theorem 3—namely $SRP([\omega_2]^\omega)$ plus $TP(\omega_2)$—can be weakened to obtain the following theorem (note that $SRP([\omega_2]^\omega)$ implies $RP([\omega_2]^\omega)$ and saturation of NS_{ω_1}; see Jech [6]):

Theorem 8. *Assume $RP([\omega_2]^\omega)$, NS_{ω_1} is saturated, and $TP(\omega_2)$ holds. Then there is no presaturated tower concentrating on GIS_{ω_1}.*

The proof of Theorem 8—like the original proof of Theorem 3—uses the very nice result of Velickovic [9] (improving on a theorem of Gitik) that implies: If W

is a transitive ZF^- model of height at least ω_2 and $\mathbb{R} - W \neq \emptyset$, then $[\omega_2]^\omega - W$ is projective stationary (Gitik's original result was the same, with the "projective" part removed). It turned out that the full power of SRP was not needed in the proof of Theorem 3; the saturation of NS_{ω_1} implies that if W is a transitive model of height ω_2 which has cofinally many internally stationary initial segments, then the stationary subsets of ω_1 which witness this internal stationarity stabilize, and this is enough (along with $RP([\omega_2]^\omega)$ in place of $SRP([\omega_2]^\omega)$) to run the argument from the proof of Theorem 3.

3. Martin's Maximum, DRP, and related topics

We also worked on problems related to "plus" versions of forcing axioms. Recall that for a class Γ of posets, $FA^{+\beta}(\Gamma)$ means that for every $\mathbb{Q} \in \Gamma$, every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{Q}$, and every β-sized collection $\mathcal{S}$ of $\mathbb{Q}$-names for stationary subsets of ω_1, then there is a filter g which meets every element of $\mathcal{D}$ and such that for all $\dot{S} \in \mathcal{S}$, $\dot{S}_g := \{\alpha < \omega_1 \mid \exists q \in g \; q \Vdash \check{\alpha} \in \dot{S}\}$ is stationary.

We worked on the following question:

Question 9. *Does Martin's Maximum imply $FA^{+\omega_1}$(σ-closed)? Or even just FA^{+2}(σ-closed)?*

This question is interesting because Shelah proved:

Theorem 10. *(Shelah [8]) Martin's Maximum implies FA^{+1}(σ-closed).*

Contrast Theorem 10 with the fact that, for example, Martin's Maximum does *not* imply PFA^{+1} (also due to Shelah [8]). In fact Velickovic [9] proved that if V is any model of MM, then one can force over V in a way which preserves MM and makes PFA^{+1} fail. The proof of Theorem 10 does not generalize even for 2 names of stationary sets (i.e. the proof does not seem to generalize to prove FA^{+2}(σ-closed) from MM.).

Ultimately we were unsuccessful in our attempts to answer Question 9. We approached the problem from a couple of directions.

On the affirmative side (i.e. trying to show MM implies $FA^{+\omega_1}$(σ-closed)), we tried to show something weaker: that MM implies the *Diagonal Reflection Principle* (DRP). DRP is a highly simultaneous version of stationary set reflection introduced in Cox [3], where it was shown that $FA^{+\omega_1}$(σ-closed) implies DRP. Cox [3] also introduced a weaker version—called $wDRP$—and showed that MM implies $wDRP$; Viale [10] independently proved a similar result. It is open whether MM implies the full DRP; we worked on this but were unsuccessful. Figure 1 on page 4 shows the known implications (RP_{ω_1} denotes the statement: "for all regular $\theta \geq \omega_2$, every ω_1-sized collection of stationary subsets of $[\theta]^\omega$ reflect simultaneously to a set of size ω_1").

On the negative side (i.e. trying to show that MM does not imply $FA^{+\omega_1}$(σ-closed)), we examined Velickovic's proof that one can force to preserve MM while forcing failure of PFA^+. Though Viale made some interesting observations about posets which preserve forcing axioms, we did not make any significant progress in this direction.

References

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FIGURE 1. MM, DRP, and $FA^+(\sigma\text{-closed})$

$$
\begin{tikzcd}
FA^{+\omega_1}(\sigma\text{-closed}) \arrow[r] \arrow[d] & FA^{+2}(\sigma\text{-closed}) \arrow[r] & FA^{+1}(\sigma\text{-closed}) \\
DRP \arrow[d] \arrow[dr] & MM \arrow[l] \arrow[ul, "?"] \arrow[u, "?"] \arrow[ur] \arrow[dl, "?"] \arrow[d] & \\
wDRP & RP_{\omega_1} &
\end{tikzcd}
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