PURPOSE OF THE VISIT

Piotr Borodulin–Nadzieja visited School of Mathematics at University of East Anglia in Norwich to work with prof. Mirna Džamonja on the project "Measure Recognition Problems".

The project concerns a set of problems motivated by the problem of classification of finitely–additive measures on Boolean algebras.

We were particularly interested in the following questions:

- problem of the complexity of the equivalence relation of measure isomorphism;
- Measure Recognition Problems (i.e. the questions of the form: how to characterize by combinatorial means those Boolean algebras which support measures with a given property, see [3]), including:
  - the characterization of Boolean algebras supporting a strictly positive separable measures;
  - the characterization of Boolean algebras supporting a strictly positive uniformly regular measures;
  - the characterization of Boolean algebras which does not carry a non–separable measures;
  - the characterization of Boolean algebras which does not carry a uniformly regular measures.

DESCRIPTION OF THE WORK CARRIED OUT DURING VISIT AND THE MAIN RESULTS OBTAINED

We call a measure μ on a Boolean algebra $\mathfrak{A}$ isomorphic to a measure ν defined on a Boolean algebra $\mathfrak{B}$ if there is a Boolean isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ which preserves the measure. We will say that a Boolean algebra *carries* a measure μ if μ is defined on $\mathfrak{A}$. If, additionally, μ is strictly positive on $\mathfrak{A}$, we will say that $\mathfrak{A}$ *supports* μ.

A measure on a Boolean algebra is *non–atomic* if for every $\varepsilon > 0$ there is a finite partition of 1 into elements of measures at most ε.

We say that a family $\mathcal{D} \subseteq \mathfrak{A}$ is *μ–dense*, if

$$\inf\{\mu(A \triangle D)\colon D \in \mathcal{D},\ A \in \mathfrak{A}\} = 0.$$

A family $\mathcal{D} \subseteq \mathfrak{A}$ is *uniformly μ–dense* if

$$\inf\{\mu(A \setminus D)\colon D \in \mathcal{D},\ A \in \mathfrak{A},\ D \subseteq A\} = 0.$$

We say that the measure μ on $\mathfrak{A}$ is *separable* if there is a countable μ–dense family $\mathcal{D} \subseteq \mathfrak{A}$ and the measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable uniformly μ–dense family $\mathcal{D} \subseteq \mathfrak{A}$.

1. The complexity of the classification task.

The classification of Boolean algebras supporting a finitely–additive measure seems to be a difficult task. One of the purposes of the project was to try to understand how difficult it can be, using the machinery of the theory of complexity of equivalence relations.

To do that we considered the simplest case: strictly positive measures defined on the Cantor algebra. The isomorphism between such measures seems already to be quite a complicated equivalence relation.

Every strictly positive measure on the Cantor algebra can be coded as an element of $(0,1)^{2^{<\omega}}$. This space endowed with the standard topology is a standard Borel space. The isomorphism of measures is an equivalence relation on this space.

We have not succeeded to calculate the precise complexity of this relation but we showed that this equivalence relation is analytic complete if we relax the assumption of strict positivity of the considered measures. We obtained also other partial results in this direction.

2. Characterization of Boolean algebras supporting a uniformly regular non–atomic measure

We presented a characterization of Boolean algebras supporting a strictly positive non–atomic uniformly regular measure.

For a measure μ defined on the Cantor algebra, we define the Jordan algebra $\mathcal{J}_\mu$ by

$$\mathcal{J}_\mu = \{A \in \mathcal{C}\colon \mu_*(A) = \mu^*(A)\},$$

where $\mathcal{C}$ is the Cohen algebra, i.e. the completion of the Cantor algebra.

We showed the following result.

Theorem 1 *For every non–atomic measure μ on the Cantor algebra, the algebra $\mathcal{J}_\mu$ is isomorphic to $\mathcal{J}_\lambda$, where λ is the standard measure on the Cantor algebra.*

We will call $\mathcal{J}_\lambda$ the Jordan algebra. This result can be seen as an analogue of the Maharam theorem saying that every Boolean algebra supporting a separable non–atomic σ–additive measure is isomorphic to the Random algebra. In our case every uniformly regular non–atomic strictly positive measure defined on a "maximal possible" Boolean algebra is isomorphic to a Lebesgue measure on the Jordan algebra. The Jordan algebra can be seen as the algebra as close to completeness as possible without loosing the property of supporting a uniformly regular measure.

We are now ready to state our characterization.

Theorem 2 *Every Boolean algebra supporting a uniformly regular non–atomic measure is isomorphic to a subalgebra of the Jordan algebra containing the Cantor algebra.*

We hope that methods used in the proof of the above theorem will find some further applications in solving some problems concerning the existence of strictly positive uniformly regular measure.

3. Other measure recognition problems

In [2] it was shown that every Boolean algebra carries either a non–separable measure or a uniformly regular one. We succeded to show that the above result cannot be strengthened by replacing *the non–existence of non–separable measure* with *the existence of ω_1 independent sequence*. Namely,

Theorem 3 *Under CH there is a Boolean algebra without a ω_1 independent sequence and without a uniformly regular measure.*

The Talagrand's construction (see [4]) serves as an example. Therefore, the result from [2] seems to reveal a real connection between the existence of a non–separable measures and uniformly regular ones.

We hoped that this result has a counterpart in the case of strictly positive measures. Then, using the theorem 2 we could say something about the existence of strictly positive non–separable measure. This turned out to be not true and the following theorem was proved using a Bell's construction of a separable space without a countable π–base (see [1]).

Theorem 4 *There is a Boolean algebra supporting a measure but which does not support either a non–separable measure or a uniformly regular one.*

References

[1] M. G. Bell, *G_κ subspaces of hyadic spaces*, PAMS **104**, No. 2 (1988), 635–640;

[2] P. Borodulin–Nadzieja, *On measures on minimally generated Boolean algebras*, Topology Appl. **154** (2007) 3107–3124;

[3] M. Džamonja, *Measure Recognition Problem*, Philosophical Transaction of the Royal Society **364**, 2006;

[4] M. Talagrand, *Un nouveau C(K) qui possede la proprit de Grothendieck*, Israel J. Math. **37** (1980), pp. 181191.

FUTURE COLLABORATION WITH HOST INSTITUTION

The collaboration between Mirna Džamonja and Piotr Borodulin–Nadzieja on the classification of Boolean algebras with finitely–additive mesures will continue at least until Summer 2010.

The visit established further connections between logic groups in Norwich and Wrocław. Francesco Piccoli and Omar Selim, the students of Mirna Džamonja from University of East Anglia, will visit Wrocław University, in February 2010 and May 2010, respectively.

PROJECTED ARTICLES RESULTING OR TO RESULT FROM THE GRANT

We are currently working on the article *On classification of measures on Boolean algebras* (P. Borodulin–Nadzieja, M. Džamonja) with the results presented above (which will also, hopefully, contain the results we will obtain in the nearest future). We plan to submit this article around June 2010. A preliminary version of the preprint will be available in March 2010 on the page http://www.math.uni.wroc.pl/~pborod/.