For two weeks (July 2-July 14, 2012), Ekaterina Fokina, Sy-David Friedman, Julia Knight, Russell Miller, and Antonio Montalbán worked at the Kurt Gödel Research Center in Vienna on problems in computable structure theory. Knight, Miller, and Montalbán were all supported by grants from the ESF. Knight and Miller continued working together at the KGRC for a few days afterwards (July 15-16) before leaving Vienna.

1 Problems and results

1.1 Problems related to Vaught's Conjecture

In [6], Goncharov and Knight asked various questions about computable structures of high Scott rank. The more recent paper of Fokina, Friedman, Harizanov, Knight, McCoy, and Montalbán [4] gives some further, related questions. Here are two.

Question 1. *Is there a computable infinitary sentence σ with (up to isomorphism) just one computable model of Scott rank $\geq \omega_1^{CK}$?*

Question 2. *Is there a computable infinitary sentence σ with (up to isomorphism) just one (or only finitely many) models $\mathcal{A}$ such that the set $I(\mathcal{A})$ of indices for computable copies of $\mathcal{A}$ is not hyperarithmetical?*

Let K, K' be "nice" classes of countable structures, closed under isomorphism, where a class is "nice" if it is axiomatized by a computable infinitary sentence.

Definition 1. *$K \leq_{tc} K'$ if there is a Turing operator $\Phi = \varphi_e$ such that for each $\mathcal{A} \in K$, there is some $\mathcal{B} \in K'$ such that*

1. *$\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$ (we write $\Phi(\mathcal{A}) = \mathcal{B}$), and*
2. *for $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$.*

If we restrict our attention to computable structures in K, and identify the structures with their indices, then the isomorphism relation becomes a Σ_1^1 equivalence relation on the index set $I(K)$. If we add an equivalence class for the numbers that are not in $I(K)$, then we obtain a Σ_1^1 equivalence relation $E(K)$ on ω.

Definition 2. *For Σ_1^1 equivalence relations E, E' on ω, $E \leq_{FF} E'$ if there is a computable function f such that mEn iff $f(m)E'f(n)$.*

In [4], we showed that for the class of trees, and for torsion-free Abelian groups, and even Abelian p-groups, $E(K)$ lies on top among Σ_1^1 equivalence relations on ω, under $\leq_{FF}$. By contrast, the class of Abelian p-groups does not lie on top under $\leq_{tc}$ (or under $\leq_B$), and it is not known whether the class of torsion-free Abelian groups lies on top under $\leq_{tc}$ (or $\leq_B$).

Question 3. *Is there a nice class K (axiomatized by a computable infinitary sentence) such that $E(K)$ is not hyperarithmetical and not on top under $\leq_{FF}$?*

We may relativize Questions 1, 2, and 3 to a set X.

Question 4 (Relativization of Question 1)**.** *Is there an X-computable infinitary sentence with (up to isomorphism) just one X-computable model of Scott rank $\geq \omega_1^X$?*

Question 5 (Relativization of Question 2)**.** *Is there an X-computable infinitary sentence with (up to isomorphism) just one model $\mathcal{A}$ such that $I^X(\mathcal{A})$ is not X-hyperarithmetical?*

Question 6 (Relativization of Question 3)**.** *Is there a class K, axiomatized by an X-computable infinitary sentence, such that $E^X(K)$ is not hyperarithmetical and not on top under $\leq_{FF}^X$ (where $\leq_{FF}^X$ is the analogue of $\leq_{FF}$ with X-computable reductions)?*

Definition 3. *A "counterexample to Vaught's Conjecture" is a sentence σ of $L_{\omega_1\omega}$ such that for any countable fragment L_A containing σ, the models of σ satisfy only countably many different complete L_A theories, and realize only countably many complete L_A types (all together), and there is no countable fragment L_A in which all of the models are atomic.*

We spent a good deal of time showing that if Vaught's Conjecture fails, then for suitably chosen X, Questions 4, 5, and 6 all have a positive answer. It turns out that the results we proved are all contained in a new paper of Becker [1]. The history is as follows. In January, Becker gave a talk in Vienna [1], with a positive result on Question 5. At the end of the talk, Friedman asked about Question 4, and Becker said that the same ideas would give a positive result for that question. In February, in Oberwolfach, Montalbán and Knight arrived at an approach to Question 4. Montalbán went on to prove, in [8], under the assumption of Projective Determinacy, that two statements in computability theory are equivalent to Vaught's Conjecture. Our proofs are quite different from Becker's. The result on Problem 4 was arrived at independently (but later) by Knight and Montalbán, and the proofs that we arrived at as a group for the other two problems grew out of that one.

The results on Questions 4, 5, and 6 are properly Becker's. We learned a great deal by working through Becker's results in our own way, and discussing the results of Montalbán. We hope to use what we learned to obtain further results in this direction.

1.2 Finitary and Countable Reducibility

During the visit, Miller gave a talk at the KGRC on his joint work [7] with Keng Meng Ng regarding finitary reducibility. This work arises from the notion of computable reducibility (also known as FF-reducibility, or m-reducibility), which is an adaptation of Borel reducibility to the setting of equivalence relations on ω. Fokina, Friedman, and Knight have all had significant experience with this concept, which has recently come to be widely studied. Miller and Ng considered a version in which one does not attempt to give a full reduction from one equivalence relation E to another one F; instead, one tries to give a uniform way of mapping each n-tuple from the field of E to an n-tuple from the field of F, so that the restriction of the relation E to the first n-tuple is isomorphic to the restriction of F to the second. In the work of Miller and Ng, ω is the field of both E and F, with the uniform map being a computable function, and there are some intriguing results which arise from this definition.

At the end of the talk, Miller proposed asking the analogous questions about Borel reducibility on equivalence relations on 2^ω: can there be such finitary reducibilities between such equivalence relations E and F (where the uniform maps on n-tuples are now Borel functions) even when there is no full Borel reduction from E to F? Additionally, in this context it makes sense to ask about reductions uniform on arbitrary countable subsets of 2^ω, giving the notion of countable Borel reducibility.

Here is the definition of the finitary and countable reducibilities.

Definition 4. *Let E, F be equivalence relations on ω. Then $E \leq_c^n F$ if there is a computable function $f : \omega^n \to \omega^n$ such that whenever $f(i_1, \ldots, i_n) = (j_1, \ldots, j_n)$, then $i_s E i_t$ iff $j_s F j_t$. If such functions can be given uniformly for all n, then E is* finitarily reducible *to F, written $E \leq_c^{<\omega} F$.*

Likewise, if E and F are equivalence relations on 2^ω, we say that $E \leq_B^n F$ if there is a Borel function from $(2^\omega)^n$ to $(2^\omega)^n$ with the same property; similarly for finitary Borel reducibility. And if $\oplus_s x_s$ represents the set $\{\langle s, n\rangle : n \in x_s\}$, then E is countably Borel reducible *to F, $E \leq_B^\omega F$, if there is a Borel function $f : 2^\omega \to 2^\omega$ such that, whenever $f(\oplus_s x_s) = \oplus_s y_s$, then $x_s E x_t$ iff $y_s F y_t$.*

In [4], it was shown that isomorphism on computable members of a class K lies on top under $\leq_c$ iff the structures in K code an ω-sequence of ordinals or ∞, which we represent by an ω-sequence of “rank-saturated trees”. In the same way, it seems that a class K lies on top under $\leq_c^n$ iff the structures in K code an n-sequence of ordinals or ∞.

Also, at the KGRC, we showed that there are Σ_1^1 equivalence relations E on 2^ω, such as isomorphism on computable graphs, which are not Borel reducible to the relation F of isomorphism on abelian p-groups, but which do have countable Borel reductions to F. Indeed, the countable reduction is continuous. So these concepts do give new insight into our study of the isomorphism relation on models of different theories. We conjecture that there are other natural relations for which certain of the finitary reducibilities hold and others (for larger arities) do not.

1.3 Categoricity

Toward the end of our time at the KGRC, we also considered the concept of the *categoricity spectrum*. The categoricity spectrum of a computable structure $\mathcal{S}$ is the set of those Turing degrees $\boldsymbol{d}$ such that $\mathcal{S}$ is $\boldsymbol{d}$-computably categorical.

Recent works [3, 5] have explored this idea to some extent, but a number of questions remain open. We focused on the Harrison ordering $\mathcal{L}$, since little work has been done on categoricity spectra of structures of high Scott rank. It is known that there exists a computable structure $\mathcal{U}$ which is universal for categoricity spectra, in the sense that its categoricity spectrum is exactly the intersection of all categoricity spectra of computable structures, and we conjecture that $\mathcal{L}$ may have the same categoricity spectrum as this $\mathcal{U}$, i.e. that $\mathcal{L}$ may also be universal for categoricity spectra. Essentially this would mean that computing isomorphisms among copies of $\mathcal{L}$ would be at least as difficult as it is for any other computable structure. We know that the categoricity spectrum of $\mathcal{L}$ is a Π^1_1 set of reals, but not Σ^1_1, and contains no hyperarithmetic set. (Hence, by a result from [3], it has no least degree.) In fact, each set in the categoricity spectrum of $\mathcal{L}$ computes every hyperarithmetical set. Moreover, there is no hyperarithmetical reduction to this categoricity spectrum from the set **WO** of well-orders of ω, since **WO** contains computable elements, whose images would have to be hyperarithmetic. All this information is consistent with $\mathcal{L}$ being universal for categoricity spectra, but none of it yet constitutes a proof. We consider this question to be worthy of further study.

References

[1] H. Becker, "Isomorphism of computable structures and Vaught's Conjecture", Talk at Kurt Gödel Research Center in Vienna, January, 2012, notes taken by Sy-David Friedman.

[2] H. Becker, "The topological Vaught's Conjecture and minimal counterexamples", *J. Symb. Logic*, vol. 29(1994), pp. 757-784.

[3] B. Csima, J. Franklin, and R. Shore, "Degrees of categoricity and the hyperarithmetic hierarchy," to appear in the *Notre Dame Journal of Formal Logic.*

[4] E. Fokina, S-D. Friedman, V. Harizanov, J. F. Knight, A. Montalbán, C. McCoy, "Isomorphism relations on computable structures", *J. Symb. Logic*, vol. 77(2012), pp. 122-132.

[5] E. Fokina, I. Kalimullin, and R. Miller, "Degrees of categoricity of computable structures," *Archive for Mathematical Logic* vol. 49 (2010) 1, pp. 51-67.

[6] S. S. Goncharov and J. F. Knight, "Structure and non-structure theorems", *Algebra and Logic*, vol. 41(2002), pp. 351-373.

[7] R. Miller and K.M. Ng, "Finitary reducibilities on equivalence relations," in preparation.

[8] A. Montalbán, "When hyperarithmetic is recursive", pre-print.