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SCIENTIFIC REPORT ON A SHORT VISIT OF PROFESSOR MARTIN–LÖF TO LEEDS
PROPOSAL TITLE: **PHILOSOPHY OF CONSTRUCTIVE SET THEORY AND TYPE THEORY**
APPLICATION REFERENCE NUMBER: 5313

1. Purpose of the visit

The main purpose of the short visit of Professor Martin–Löf to Leeds was to foster the use of constructive set theory, in the form of Martin–Löf type theory [17, 19], as a tool for and an environment within which to carry out philosophical reflection. In so doing, the very notion of set which is characterised by constructive type theory becomes more perspicuous. The visit has, in fact, further witnessed the philosophical vocation of Martin–Löf type theory. Type theory has the peculiarity of being both a mathematical system, describing a notion of constructive set, and having a well thought out philosophical justification [18, 20]. A key feature of this system is that the notion of set is intimately related to the notion of proposition due to the so called Curry–Howard correspondence which is at the heart of type theory; as a consequence an analysis of the notion of set arising in type theory is simultaneously also a contribution to the analysis of the notion of proposition, which features prominently within philosophical discussions on language and ontology. In fact, this set theory precisely characterizes and thus clarifies fundamental philosophical notions such as that of proposition, judgement and truth.

2. Description of the work carried out during the visit and of the main results obtained

Type theory has had a double role in the investigations carried out in preparation for and during this visit:

- As instrument within which to carry out philosophical elucidation of complex, central issues in philosophy, including metaphysics. That is, as a form of "mathematical philosophy".
- As a context within which to clarify the philosophy of constructive mathematics and the relation between classical and constructive practices. That is, as an environment for the development of a "philosophy of (constructive) mathematics".

As to the first point, Per Martin–Löf has elaborated and discussed a completely new account within type theory of the notion of truth for empirical propositions. The starting point for Martin–Löf's investigation was Dummett's careful analysis of statements about the past in [13, 14], and his reflection on the relation that this bears with the realism versus anti–realism debate. In his treatment Martin–Löf makes full use of the flexibility of the language of type theory, which is a dependent type theory, thus featuring dependence on a context. According to type theory, the truth of a purely mathematical proposition is given by our being in possession of a proof of it; in the case of empirical propositions, however, there is a further dependence from a measurement or experiment, which can be naturally expressed within a type–theoretic language by use of the context. Martin–Löf's type–theoretic analysis essentially agrees with Dummett's conclusions and extends it to general empirical statements (that is, not only about the past), by showing the failure of the disjunction property in such contexts. That is, contrary to the case of pure constructive mathematical statements, for empirical ones we can not hope to have in general that if $A \vee B$ is true then either A is true or B is true. A fascinating link is thus built between a mathematical theory which simultaneously analyses the notion of set and that of proposition, and a philosophical analysis of empirical statements, where the mathematical theory furnishes a more detailed and precise characterisation of the philosophical analysis. This work is highly original and is bound to rise further discussions and study within the philosophical community.

The other role of constructive type theory was as a framework within which to carry out a philosophy of constructive mathematics. Constructive mathematics has recently received renewed attention within the philosophy of mathematics, where it is has been taken as example of a variety of mathematics distinct from the classical one, in order to support forms of pluralism [12, 5, 24]. This can be seen as a positive development within the philosophy of mathematics, especially as it seems to witness a new tendency to broaden the focus of the philosophical discussion from the restricted issue of the use of the intuitionistic versus the classical *logic*, to an analysis of the *mathematics* in its entirety (based on intuitionistic or classical logic). It also highlights a very positive new trend within the philosophy of mathematics: to pay more attention to the mathematical practice [16].

The relation between constructive and classical mathematics is at the centre of reflections by constructive mathematicians on their practice [6, 7, 8, 9, 10, 21, 22, 23]. These reflections distinguish themselves from most of the philosophical literature on the topic, as they stress the *compatibility* of classical and constructive mathematics, rather then highlighting their *conflict*. The constructive mathematicians, for example, stress that every theorem in constructive mathematics is a theorem of classical mathematics, and that classical mathematics is a powerful source of inspiration for the constructive mathematician. In fact, constructive mathematics can be seen as a *generalization* of classical mathematics [21]. The main difference between classical and constructive mathematics lies not in their philosophical orientation (e. g. in favour of realism or anti–realism) but in their *methodology*: constructive mathematics has more demanding standards of proofs, requiring, for example, that a proof of an existential statement exhibits its witness and that a proof of a disjunction enables us (in principle) to determine which of the two disjuncts holds. This, in turn, ensures that constructive mathematical statements have a direct computational content, and thus also a more concrete meaning compared with general classical statements.

It is tempting to start from these remarks by constructive mathematicians to essay a conciliatory reading of the relation between constructive and classical mathematics, as it seems to better comply with the mathematical practice itself as well as with the way constructive foundational systems are usually formulated. For example, in the case of Constructive Zermelo Fraenkel set theory, CZF [1, 2, 3], which in the light of Aczel's interpretation in type theory can be seen as a particular variant of type theory, the only addition of the principle of excluded middle gives all classical ZF. However, making good philosophical sense of a conciliatory view of this kind which is also faithful to the constructive attitude more at large, has turned out to be very difficult. In particular, it requires a careful analysis of the characteristics of each of these mathematical traditions, including the differences between them, addressing, for example, difficult issues of meaning of mathematical statements and the notions of classical and constructive proofs. The visit by Professor Martin–Löf has been extremely useful in making some first attempts at a general characterization of a possible philosophy of constructive mathematics which embraces a conciliatory attitude to the "conflict" between constructive and classical mathematics without receding to a form of pluralism. The main outcome of the visit at this very preliminary stage has been to clarify which possible routes such a view could take and compare them with the philosophy underlying type theory.

3. Future collaboration with host institution

The visit has helped setting up some targets for future research, as it has clarified that type theory can be seen as having, again, a double role. It can play a similar role as Bishop's mathematics, being a base for a comparison between the classical and the intuitionistic traditions, especially clarifying their fundamental notions of *set*. And it can be used as a tool for analysing the feasibility of a philosophical position as suggested above. Type theory, in fact, endows its constructive notion of set with direct computational meaning [18, 20], as witnessed by the fundamental role it is having for the development of the area of program extraction from constructive proofs [4, 11, 15]. In particular, compared with informal presentations of Bishop style mathematics, it has the advantage of allowing for a precise, formal comparison between the classical and constructive notions of proof and of the meaning of mathematical statements.

The hope is that this short visit has started a fruitful collaboration between Leeds and Stockholm on the philosophy of constructive mathematics and on constructive type theory as a philosophical tool.

During the visit Per Martin–Löf has exchanged ideas with other members of the Leeds logic group, like Peter Schuster, Michael Rathjen and Nicola Gambino, as well as Peter Aczel (Manchester) and numerous philosophers, including Crispin Wright (Aberdeen and New York) and Ian Rumfitt (London), who were also visiting Leeds.

Per Martin–Löf has given a public lecture on "Truth of Empirical Propositions" on 4th of September 2013 which has sparked a lively discussion. It was attended by philosophers as well as mathematicians, including a high number of PhD students.

4. Projected publications

Laura Crosilla has presented some thoughts on the philosophy of constructive mathematics, which have highly benefitted from discussions with Per Martin–Löf during his visit, at the meeting "Proof" in Bern, 9-13 September 2013. She is in the process of writing an article on this topic.

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