ESF - Short Visit Grant - Final Report

Purpose of the visit

I applied for the Short Visit Grant to conduct joint research with prof. Joerg Brendle in Bonn. We had started to prepare a joint paper about generic existence of certain classes of ultrafilters and the purpose of the visit was to accomplish the work.

Description of the work carried out during the visit

In the first day of my visit I presented some results of my former research in the seminar of Bonn Logic Group and for the rest of the week I discussed with prof. Joerg Brendle topics related to our joint paper about $\mathcal{I}$-ultrafilters, which we are finalizing.

Our research is focused on $\mathcal{I}$-ultrafilters which were introduced by Baumgartner: Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on ω, we say that $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for any $F : \omega \to X$ there is $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$. We are mainly interested in $\mathcal{I}$-ultrafilters for $X = \omega$ and $\mathcal{I}$ an ideal on ω. These are the ideals, on which we focused our attention:

The density ideal $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A\cap n|}{n} = 0\}$

The summable ideal $\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n} < \infty\}$

The ideal $\mathcal{I}_{SC}$ generated by SC-sets where $A \subseteq \mathbb{N}$ is called an SC-set if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$ for the increasing enumeration $A = \{a_n : n \in \omega\}$

The ideal $\mathcal{I}_T$ generated by thin sets where $A \subseteq \mathbb{N}$ is called thin if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$ for the increasing enumeration $A = \{a_n : n \in \omega\}$

Ultrafilters from a given class $\mathcal{C}$ *exist generically* if each filter base of size less than $\mathfrak{c}$, the cardinality of continuum, can be extended to an ultrafilter belonging to $\mathcal{C}$. Generic existence of ultrafilters can be characterized in terms of cardinal equalities. Given a class of ultrafilters $\mathcal{C}$ let us introduce cardinal invariant $\mathfrak{ge}(\mathcal{C})$, which denotes the minimal cardinality of a filter base which cannot be extended to an ultrafilter from $\mathcal{C}$. It is obvious that ultrafilters from $\mathcal{C}$ exist generically if and only if $\mathfrak{ge}(\mathcal{C}) = \mathfrak{c}$.

If $\mathcal{C}$ is a class of $\mathcal{I}$-ultrafilters for a given ideal $\mathcal{I}$ on ω we write $\mathfrak{ge}(\mathcal{I})$ instead of $\mathfrak{ge}(\mathcal{I}$-ultrafilters). One can observe that

$$\mathfrak{ge}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ filter base}, \mathcal{F} \subseteq \mathcal{I}^+ \wedge (\forall A \in \mathcal{I})(\exists F \in \mathcal{F})|A \cap F| < \omega\}$$

In some cases the value $\mathfrak{ge}(\mathcal{C})$ coincides with some known cardinal invariants e.g. $\mathfrak{ge}(P\text{-points}) = \mathfrak{d}$ [Ketonen], $\mathfrak{ge}(\text{selective ultrafilters}) = \text{cov}(\mathcal{M})$ [Canjar], $\mathfrak{ge}(\text{nowhere dense ultrafilters}) = \text{cof}(\mathcal{M})$ [Brendle].

The aim of our joint work with prof. Joerg Brendle is to determine (if possible) the value of $\mathfrak{ge}(\mathcal{I})$ for the ideals on ω mentioned above with the means of cardinal invariants of the continuum studied in the literature and/or cardinal invariants associated to the ideal $\mathcal{I}$.

The following two cardinal invariants associated to an ideal $\mathcal{I}$ are of special importance in this context:

$\text{non}^*(\mathcal{I}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq [\omega]^\omega, (\forall A \in \mathcal{I})(\exists X \in \mathcal{X})\, |A \cap X| < \omega\}$

$\text{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})\, I \subseteq A\}$

It is not difficult to check that $\text{non}^*(\mathcal{I}) \leq \mathfrak{ge}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ for every ideal $\mathcal{I}$.

We investigated cardinals $\text{non}^*(\mathcal{I})$ and $\text{cof}(\mathcal{I})$ for several ideals on ω and considered also values of cardinal invariants

$\text{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}, (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})\, I \subseteq A\}$

for several pairs of ideals $\mathcal{I} \subseteq \mathcal{J}$ because of the following characterization of $\mathfrak{ge}(\mathcal{I})$:

$$\mathfrak{ge}(\mathcal{I}) = \min\{\text{cof}(\mathcal{I}, \mathcal{J}) : \mathcal{I} \subseteq \mathcal{J}\} = \min\{\text{cof}(\mathcal{J}) : \mathcal{I} \subseteq \mathcal{J}\}$$

We looked for other possible lower and upper bounds for $\mathfrak{ge}(\mathcal{I})$ for several ideals $\mathcal{I}$. Using the forcing method we constructed several models demonstrating strict inequalities in many of the discovered estimations.

Description of the main results obtained

About the density ideal $\mathcal{Z}_0$: Whereas it is known that $\text{cof}(\mathcal{Z}_0) = \text{cof}(\mathcal{N})$, $\text{non}^*(\mathcal{Z}_0)$ is not very well understood yet. We were able to prove that consistently $\mathfrak{ge}(\mathcal{Z}_0)$ differs from both the lower and upper bound and we found also other lower bounds

- CON($\mathfrak{ge}(\mathcal{Z}_0) < \mathrm{cof}(\mathcal{Z}_0)$)
- CON($\mathfrak{ge}(\mathcal{Z}_0) > \mathrm{non}^*(\mathcal{Z}_0)$)
- $\mathfrak{ge}(\mathcal{Z}_0) \geq \max\{\mathrm{cov}(\mathcal{N}), \mathrm{non}^*(\mathcal{Z}_0), \mathfrak{d}, \mathrm{non}(\mathcal{E})\}$
  where $\mathfrak{d}$ is the dominating number and $\mathcal{E}$ is the σ-ideal on $\mathbb{R}$ generated by closed measure zero sets.

About the summable ideal $\mathcal{I}_{1/n}$: The obvious lower and upper bound are again consistently distinct from $\mathfrak{ge}(\mathcal{I}_{1/n})$. As in the case of density ideal, $\mathrm{cov}(\mathcal{N})$ is a lower bound for $\mathfrak{ge}(\mathcal{I}_{1/n})$, however, $\mathfrak{d}$ is not.

- CON$\big(\mathfrak{ge}(\mathcal{I}_{1/n}) < \mathrm{cof}(\mathcal{I}_{1/n})\big)$
- CON($\mathfrak{ge}(\mathcal{I}_{1/n}) > \mathrm{non}^*(\mathcal{I}_{1/n})$)
- $\mathfrak{ge}(\mathcal{I}_{1/n}) \geq \max\{\mathrm{cov}(\mathcal{N}), \mathrm{non}^*(\mathcal{I}_{1/n}))\}$
- CON($\mathfrak{ge}(\mathcal{I}_{1/n}) < \mathfrak{d}$)

About the ideal $\mathcal{I}_{SC}$: Since this ideal has not been studied so carefully in the past as the first two, we started by investigation of the associated cardinal invariants $\mathrm{cof}(\mathcal{I}_{SC})$ and $\mathrm{non}^*(\mathcal{I}_{SC})$. We determined exact value for the first one and we found an upper bound for the second one. We provided models for strict inequality between $\mathfrak{ge}(\mathcal{I}_{SC})$ and $\mathrm{cof}(\mathcal{I}_{SC})$ and another model for strict inequality between $\mathfrak{ge}(\mathcal{I}_{SC})$ and $\mathrm{non}^*(\mathcal{I}_{SC})$. The dominating number $\mathfrak{d}$ is a lower bound for $\mathfrak{ge}(\mathcal{I}_{SC})$ and $\mathrm{cov}(\mathcal{N})$ is not.

- $\mathrm{cof}(\mathcal{I}_{SC}) = \mathfrak{c}$, $\mathrm{non}^*(\mathcal{I}_{SC}) \leq \mathfrak{r}$ where $\mathfrak{r}$ is the reaping number
- CON($\mathfrak{ge}(\mathcal{I}_{SC}) < \mathrm{cof}(\mathcal{I}_{SC})$)
- CON($\mathfrak{ge}(\mathcal{I}_{SC}) > \mathrm{non}^*(\mathcal{I}_{SC})$)
- $\mathfrak{ge}(\mathcal{I}_{1/n}) \geq \max\{\mathfrak{d}, \mathrm{non}^*(\mathcal{I}_{SC}))\}$
- CON($\mathfrak{ge}(\mathcal{I}_{SC}) < \mathrm{cov}(\mathcal{N})$)

About the ideal $\mathcal{I}_T$: First we considered the cardinal invariants $\mathrm{cof}(\mathcal{I}_T)$ and $\mathrm{non}^*(\mathcal{I}_T)$. We showed that the generic existence of thin ultrafilters is equivalent to the generic existence of Q-points and we proved that consistently $\mathrm{cof}(\mathcal{I}_T)$ is strictly greater than $\mathfrak{ge}(\mathcal{I}_T)$. In contrast to all the previous ideals we were not able to construct a model, in which $\mathfrak{ge}(\mathcal{I}_T) > \mathrm{non}^*(\mathcal{I}_T)$ and we conjectured that the cardinals are equal in ZFC. Since $\mathcal{I}_T$ is contained in both $\mathcal{I}_{SC}$ and $\mathcal{I}_{1/n}$ (hence the corresponding classes of $\mathcal{I}$-ultrafilters are in inclusion), neither $\mathrm{cov}(\mathcal{N})$, nor $\mathfrak{d}$ are lower bounds for $\mathfrak{ge}(\mathcal{I}_T)$. We proved that $\mathrm{non}(\mathcal{N})$ is upper bound for $\mathfrak{ge}(\mathcal{I}_T)$ and constructed a model, in which the inequality is strict.

- $\mathrm{cof}(\mathcal{I}_T) = \mathrm{cof}(\mathcal{I}_T, \mathcal{I}_{SC}) = \mathfrak{c}$, $\mathrm{non}^*(\mathcal{I}_T) \leq \mathfrak{r}$ where $\mathfrak{r}$ is the reaping number, $\mathrm{CON}(\mathrm{non}^*(\mathcal{I}_T) < \mathrm{non}^*(\mathcal{I}_{SC}))$
- $\mathfrak{ge}(\mathcal{I}_T) = \mathfrak{ge}(Q\text{-points})$
- $\mathrm{CON}(\mathfrak{ge}(\mathcal{I}_T) < \mathrm{cof}(\mathcal{I}_T))$
- $\mathrm{CON}(\mathfrak{ge}(\mathcal{I}_T) < \mathrm{cov}(\mathcal{N}))$ and $\mathrm{CON}(\mathfrak{ge}(\mathcal{I}_T) < \mathfrak{d})$
- $\mathfrak{ge}(\mathcal{I}_T) \leq \mathrm{non}(\mathcal{N})$
- $\mathrm{CON}(\mathfrak{ge}(\mathcal{I}_T) < \mathrm{non}(\mathcal{N}))$

Future collaboration with host institution

Since prof. Joerg Brendle is not a permanent member of the Mathematical Logic Group at the University of Bonn and his sabbatical leave in Bonn ends in September no plans for future collaboration with University of Bonn exist.

Projected publications/articles resulting or to result from your grant

An article containing the main results we obtained during my visit in Bonn is in preparation. We expect that the paper with the title "Generic existence of $\mathcal{I}$-ultrafilters" (or similar) will be ready for submission by the end of the year 2009.