ESF Research Networking Programmes

Exchange Grant No. 4143

Scientific Report

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The present report is with regard to the INFTY network exchange grant No. 4143, which I received for a research stay at the University of East Anglia (UEA) in Norwich, UK. The visit started on 18/02/2013 and lasted for seven (7) weeks, until 08/04/2013.

1. Purpose of the visit

The purpose of the visit was to conduct mathematical research in the area of set theory, jointly with Dr. David Asperó who is a lecturer at the department of mathematics of UEA. In particular, our plan was to work on the so−called *resurrection axioms*, which are strong forms of forcing axioms that were introduced in unpublished work by J.D. Hamkins and T. Johnstone (cf. [5]).

I had already studied variants of such axioms in my doctoral dissertation (cf. [7]) and, moreover, I had introduced even stronger versions of them, under the general name *unbounded resurrection axioms.* There are several open questions in this area and the aim of the research visit was to investigate further such principles.

2. Work carried out & Main results

There were two main themes in the work which was carried out during the research visit. On the one hand, we studied versions of the unbounded resurrection axioms, as it was initially intended. On the other hand, and of a significantly different flavour, we introduced new sorts of "real numbers" which we call *long reals*; these are complete ordered fields which arise via a natural modification of the well−known construction which produces the standard field $\mathbb{R}$ from the set of natural numbers $\mathbb{N}$.

Both themes will be explained below in more detail. In what follows, all uncredited definitions and results are joint work of David Asperó and the author, carried out during the author's visit to UEA.

2.1. **Resurrection axioms.** Before describing the work done and the results obtained during the visit, let us first recall the relevant concepts. We start from the definition of the resurrection axiom(s), as introduced by Hamkins and Johnstone.

Definition 1 ([5])**.** For any given (definable) class Γ of posets, the **Resurrection Axiom** for Γ, denoted by $\mathsf{RA}(\Gamma)$, is the assertion that for any $\mathbb{Q} \in \Gamma$, there exists a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$" and $H_{\mathfrak{c}} \prec (H_{\mathfrak{c}})^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$ (where $\mathfrak{c} = 2^{\aleph_0}$).

The unbounded forms of the resurrection axioms were introduced in my doctoral dissertation, as follows.

Date: April 17, 2013.

Definition 2 ([7]). For any given (definable) class Γ of posets, the **Unbounded Resurrection Axiom** for Γ, denoted by $\mathsf{UR}(\Gamma)$, is the assertion that for every cardinal $\beta > max\{\omega_2\,, \mathfrak{c}\}$ and every $\mathbb{Q} \in \Gamma$ with $\mathbb{Q} \in H_\beta$, there is a $\mathbb{Q}$ - name for a poset $\dot{\mathbb{R}}$ with $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding

$$j : H_\beta \longrightarrow (H_{j(\beta)})^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$$

such that $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $cp(j) = max\{\omega_2\,, \mathfrak{c}\}$ and $j(cp(j)) > \beta$.

For the classes Γ of c.c.c., σ - closed, proper, $\aleph_1$ - semiproper and of stationary preserving posets, it is shown in [7] that the consistency of the corresponding UR axiom is derived from that of the existence of an extendible cardinal. Moreover, for the cases of c.c.c. and σ - closed posets, the following (indirect) consistency lower bounds are obtained.

Theorem 3 ([7]). UR(c.c.c.) *implies that, for every* $\lambda > \mathfrak{c}$ *with* $cf(\lambda) < \mathfrak{c}$, *there is no good* λ^+ *- scale.*

Theorem 4 ([7]). UR(σ - closed) *implies that, for every (uncountable) strong limit* λ *with* $cf(\lambda) = \omega$, $\square^*_\lambda$ *fails.*

In the open questions stated at the end of [7], it was asked whether the failure of even weaker principles (such as the *approachability property* at λ; denoted by AP_λ) can be obtained in a similar way from these axioms.

With David Asperó we observed that, in the case of c.c.c. posets, the answer is "yes"; this follows from the known fact that, for singular λ, AP_λ implies that every λ^+ - scale is good (see [3]). In addition, and by appealing to the notion of a *indestructibly generically supercompact* cardinal [1], we were able to generalize Theorem 4 in the following sense.

Theorem 5. *Let* $n \in \omega$ *and suppose that* ω_{n+2} *is indestructibly generically supercompact by the class of* ω_{n+1} *- closed posets. Then, for every (uncountable) strong limit* λ *with* $cf(\lambda) = \omega_n$, $\square^*_\lambda$ *fails.*

Additionally, and for the class of stationary preserving posets, we have shown the following.

Theorem 6. *Assume* MM^{++} *and that there exists a proper class of Woodin cardinals. Then,* UR(stationary preserving) *holds.*

This result is a significant improvement – in terms of a consistency upper bound – for the axiom UR(stationary preserving). It is obtained by employing Woodin's *stationary tower* forcing (see [6]). It should be pointed out, however, that there seems to be a difference between the two methods for obtaining models of this axiom.

The initial method of forcing UR(stationary preserving) from an extendible cardinal (cf. [7]) allows for tail forcings which collapse cofinalities to ω (e.g., Prikry forcing). On the other hand, the method using Woodin cardinals and the stationary tower does not allow for such collapses of cofinalities. Whether one can draw any interesting consequences from this distinction is a matter of further study.

[1] See [2] or [7] for the definition of such a cardinal. What is actually used in the proof of Theorem 4 is the assumption that ω_2 is indestructibly generically supercompact by the class of σ - closed posets. Such an assumption follows from UR(σ - closed); see [7].

We have also spent time organizing our general knowledge on the resurrection axioms and we have, as a result, pinpointed some lines of investigation. For instance, we have looked at variant families of the UR axioms and we have identified, in the same context and as a potentially fruitful connection, the perspective of looking at generic embeddings in terms of suitable *ideals*.

While we were working on these latter issues, our attention was distracted by a quite different theme: the construction of the *long reals*, which I will describe below. We were very quickly absorbed by this fruitful – as it turns out – subject, spending a significant amount of time entirely on it. As far as the resurrection axioms are concerned, no other major result has emerged and, although we are currently preparing a comprehensive preprint on this topic, it should be understood as work still in progress.

2.2. **Long reals.** Let us first recall that there is a well – known procedure which, starting with the set of natural numbers $\mathbb{N} = \omega$, produces the complete ordered field $\langle \mathbb{R}, 0, 1, +, \cdot, < \rangle$. This is done in steps: one first constructs the ring of integers $\langle \mathbb{Z}, 0, 1, +, \cdot, < \rangle$ and, then, the ordered field of rationals $\langle \mathbb{Q}, 0, 1, +, \cdot, < \rangle$. Both of these are obtained in a natural fashion, using the commutative operations of addition and multiplication on ω. Finally, one considers the *completion* of the rationals with respect to *Cauchy sequences* (or, equivalently, using *Dedekind cuts*), resulting in the complete ordered field of real numbers.

Our initial motivation was the following question.

Question. What happens if we replace ω in the above construction by some other cardinal $\kappa > \omega$?

The first thing to notice is that the ordinary operations on the ordinals (that is, addition and multiplication) are not of much help in this context since they fail to be commutative. One solution to this problem is to use the *Hessenberg* ordinal operations $\oplus$ and $\otimes$ (which are also called *natural operations*; see, e.g., [1] or [4]). These are defined as follows, for any ordinals α and β.

For the natural sum, we let

$$\alpha \oplus \beta = max\{\alpha, \beta\} + min\{\alpha, \beta\}.$$

Equivalently, $\alpha \oplus \beta$ is the order – type of the *longest* well – order which extends the disjoint union of α and β.

To define $\otimes$, we use the *Cantor normal form* and write (uniquely) the ordinals α and β as polynomials:

$$\alpha = p_\alpha(\omega) = \omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \ldots + n_k,$$

where $k \in \omega$, $\alpha_0 > \alpha_1 > \ldots > \alpha_{k-1}$ and $n_i \in \omega$ for all $i < k+1$, and

$$\beta = p_\beta(\omega) = \omega^{\beta_0} \cdot m_0 + \omega^{\beta_1} \cdot m_1 + \ldots + m_l,$$

where $l \in \omega$, $\beta_0 > \beta_1 > \ldots > \beta_{l-1}$ and $m_i \in \omega$ for all $i < l+1$. We then let

$$\alpha \otimes \beta = p_\alpha(\omega) \cdot p_\beta(\omega),$$

where, for the latter product, we compute the formal polynomial product of $p_\alpha(\omega)$ and $p_\beta(\omega)$, and use $\oplus$ for all relevant additions. Equivalently, $\alpha \otimes \beta$ is the order – type of the *longest* well – order which extends the product order on $\alpha \times \beta$.

Clearly, $\oplus$ and $\otimes$ are commutative and associative, 0 is the identity for $\oplus$, and 1 is the identity for $\otimes$. Moreover, the distributive law holds on both sides.

Now suppose that κ is an infinite cardinal and consider the (commutative ordered) ring $\langle \kappa, 0, 1, \oplus, \otimes, < \rangle$. Using the exact same procedure which produces $\mathbb{Z}$ and $\mathbb{Q}$ from ω, we may construct the "$\kappa-$integers" (denoted by $\kappa-\mathbb{Z}$) and, then, the ordered field of the "$\kappa-$rationals" (denoted by $\kappa-\mathbb{Q}$). In this terminology, $\omega-\mathbb{Z}$ and $\omega-\mathbb{Q}$ are just the standard integer and rational numbers, respectively.

Notation. From now on, and in order to easy readability, we will write $+$ and $\cdot$ instead of $\oplus$ and $\otimes$, although we will be exclusively using the latter operations. Moreover, we sometimes drop the multiplication symbol altogether and write $\alpha\beta$ to mean $\alpha \otimes \beta$. Finally, expressions of the form $-\alpha$ and $\frac{1}{\alpha}$ have the intended meaning.

Note that, in the typical case of interest in which $\kappa > \omega$ is an uncountable cardinal, the field $\kappa-\mathbb{Q}$ contains elements of the form $\omega - 3$, $\frac{1}{\omega^2}$, etc. It is important to keep in mind that exponentiation is always understood as *ordinary ordinal exponentiation.* Although one may try to define an appropriate notion of "natural exponentiation", one which behaves in the expected way with respect to the natural product, it turns out that this is impossible, as highlighted by the next two results.

Theorem 7. *There is no function* $f : \omega \longrightarrow \mathbf{ON}$ *such that:*

(i) $f(2) \geqslant \omega$.
(ii) *For all* $n, m \in \omega$, *if* $n < m$ *then* $f(n) \leqslant f(m)$.
(iii) *For all* $n, m \in \omega$, $f(n \cdot m) = f(n) \cdot f(m)$.

Corollary 8. *Let* $\kappa > \omega$ *be a cardinal. Then, there is no function* $\exp : \kappa \times \kappa \longrightarrow \mathbf{ON}$ *such that:*

(i) *For all* $n, m \in \omega$, $\exp(n \cdot m, \omega) = \exp(n, \omega) \cdot \exp(m, \omega)$.
(ii) *For all* $n, m \in \omega$, *if* $n < m$ *then* $\exp(n, \omega) \leqslant \exp(m, \omega)$.
(iii) *For all* $n, m \in \omega$ *and for all* $\alpha < \beta < \kappa$, $\exp(n, \alpha) \leqslant \exp(n, \beta)$.

Once we have the $\kappa-$rationals (for some cardinal $\kappa > \omega$), it is only natural to wonder whether one may "complete" them in order to produce the "$\kappa-$reals". For this, we first define what it means for a sequence of $\kappa-$rationals to be Cauchy, as follows.

Definition 9. Let $\kappa > \omega$ be a cardinal and let $(a_\xi)_{\xi<\lambda}$ be a sequence of $\kappa-$rationals, for some $\lambda \leqslant \kappa$. We say that the sequence is Cauchy if, for every $\alpha < \kappa$, there exists some $\xi_0 < \lambda$ such that, for all $\xi, \xi' > \xi_0$ we have that

$$|a_\xi - a_{\xi'}| < \frac{1}{\alpha}.$$

Cauchy sequences are always bounded. On the other hand, note that, for instance, the (bounded) sequence $(\frac{1}{n+1})_{n<\omega}$ is not Cauchy in the $\kappa-$rationals, whenever $\kappa > \omega$. Still, for regular cardinals we have the following.

Proposition 10. *Suppose that* $\kappa > \omega$ *is a regular cardinal and let* $(a_\xi)_{\xi<\kappa}$ *be an increasing* (*or a decreasing*) *sequence of* $\kappa-$*rationals. Then,* $(a_\xi)_{\xi<\kappa}$ *is Cauchy if and only if it is bounded.*

Given the notion of a Cauchy sequence, we may now consider the completion of the $\kappa-$rationals with respect to such sequences, appealing to the usual construction which produces $\mathbb{R}$ from $\mathbb{Q}$. This results in the complete ordered field of the $\kappa-$reals, which we denote by $\kappa-\mathbb{R}$. In this terminology, $\omega-\mathbb{R}$ denotes the field of ordinary real numbers. A similar construction can be done via (appropriately chosen) Dedekind cuts, resulting in an isomorphic object. At any rate, the field $\kappa-\mathbb{R}$, despite its Cauchy completeness, is far from being called a "continuum" since it has many "holes":

Proposition 11. *For every $\kappa > \omega$, there is no $x \in \kappa-\mathbb{Q}$ such that*

$$\left\{\frac{n}{\omega} : n \in \omega\right\} < x < \left\{\frac{1}{n} : n \in \omega\right\}.$$

In particular, $\kappa-\mathbb{R}$ is never $\aleph_1-$saturated as a linear order.

Clearly, for every infinite κ, we have that $\kappa-\mathbb{Q} \subseteq \kappa-\mathbb{R}$. It is natural (?) to expect that this inclusion is strict, as it is the case for the standard reals, i.e., when $\kappa = \omega$. Perhaps surprisingly, this is not the case in general. In fact, we have the following characterization:

Theorem 12. *For any infinite cardinal κ, $cf(\kappa) > \omega$ if and only if $\kappa-\mathbb{Q} = \kappa-\mathbb{R}$.*

That is, when $cf(\kappa) > \omega$, the $\kappa-$rationals are already Cauchy complete and, consequently, $|\kappa-\mathbb{Q}| = |\kappa-\mathbb{R}| = \kappa$. As an interesting corollary, and using the uncountable categoricity of (the theory of) algebraically closed fields, we have the following:

Corollary 13. *The Continuum Hypothesis holds if and only if $\mathsf{alg}(\omega_1-\mathbb{Q})$ (the algebraic closure of $\omega_1-\mathbb{Q}$) is isomorphic to the field of complex numbers $\mathbb{C}$.*

Notwithstanding, for cardinals of countable cofinality the situation changes.

Theorem 14. *For any cardinal κ with $cf(\kappa) = \omega$, $|\kappa-\mathbb{R}| = \kappa^{\aleph_0}$. In particular, we have that $\kappa-\mathbb{R} \setminus \kappa-\mathbb{Q} \neq \varnothing$.*

In fact, in the (uncountable) case of $cf(\kappa) = \omega$, we can represent (uniquely) every *irrational* $\kappa-$real $x \in \kappa-\mathbb{R} \setminus \kappa-\mathbb{Q}$ by an expression of the form:

$$x = q_0 \cdot \omega^{\alpha_0} + \ldots + q_k \cdot \omega^{\alpha_k} + \sum_{i<\omega} \frac{r_i}{\omega^{\beta_i}},$$

where $k \in \omega$, $q_0, \ldots, q_k, r_i \in \omega-\mathbb{Q}$, and where $\alpha_j, \beta_i \in \kappa$ with $\alpha_0 > \ldots > \alpha_k$ and with $(\beta_i)_{i<\omega}$ being a strictly increasing sequence cofinal in κ.

It follows from the two previous theorems that, for any $\kappa > \omega$, ordinary square roots such as $\sqrt{2}$ do not exist in the field $\kappa-\mathbb{R}$. In other words, $\kappa-\mathbb{R}$ is far from being algebraically closed. On the other hand, for instance in the case of $\aleph_\omega-\mathbb{R}$, we do have plenty of irrational numbers and some (non−ordinary) square roots. For example:

$$\sum_{n<\omega} \frac{1}{\aleph_n} = \sqrt{\sum_{n \neq m < \omega} \left(\frac{1}{\aleph_n^2} + \frac{2}{\aleph_n \cdot \aleph_m}\right)}.$$

We have also looked at the calculus of the $\kappa-$reals, which turns out to be quite pathological. Although one may easily define continuity and differentiation of functions $f : \kappa-\mathbb{R} \longrightarrow \kappa-\mathbb{R}$ in a natural fashion, many well−known theorems of the standard calculus (e.g., Bolzano's theorem, Intermediate Value theorem) do not work in general. In spite of this deficit, we do have special cases in which such standard results go through and we are currently looking for nice characterizations of functional properties which actually make these theorems work. This is still work in progress.

In addition, and of a more set−theoretic flavour, we have considered the $\kappa-$reals as a forcing notion; that is, we consider the poset $\mathbb{P}^{(\kappa)}$ consisting of the (non−empty)

κ – rational intervals, ordered by inclusion. When $\kappa = \omega$, $\mathbb{P}^{(\omega)}$ is just ordinary Cohen forcing. More generally:

Theorem 15. *For any infinite cardinal κ, $\mathbb{P}^{(\kappa)}$ is forcing – equivalent to $Col(\omega, \kappa)$ (the collapse of κ to ω using finite conditions).*

Corollary 16. *For any $\kappa > \omega$, there are $\aleph_1$ – many open dense subsets of κ – $\mathbb{R}$ whose intersection is empty. In particular, κ – $\mathbb{R}$ is the union of $\aleph_1$ – many meagre sets.*

However, we can still rescue the Baire Category theorem in the case of countable cofinality.

Proposition 17. *If $cf(\kappa) = \omega$, then the intersection of countably many open dense subsets of κ – $\mathbb{R}$ is dense in κ – $\mathbb{R}$.*

Recall that a *perfect* set is a closed set without any isolated points. If $cf(\kappa) = \omega$ and $X \subseteq \kappa$ – $\mathbb{R}$ is perfect, then $|X| \geqslant 2^{\aleph_0}$. Moreover, we can prove the analogue of the Cantor – Bendixson theorem.

Proposition 18. *If $cf(\kappa) = \omega$ and $C \subseteq \kappa$ – $\mathbb{R}$ is closed, then there exists $X \subseteq C$ with $|X| \leqslant \kappa$ such that $C \setminus X$ is perfect.*

We say that $X \subseteq \kappa$ – $\mathbb{R}$ has the *perfect set property* if either $|X| \leqslant \kappa$ or X contains a perfect set. Then, we may generalize the previous proposition in order to account for the algebra of Borel sets.

Theorem 19. *If $cf(\kappa) = \omega$ then every Borel subset of κ – $\mathbb{R}$ has the perfect set property.*

The proof of the previous theorem uses *determinacy* of Borel sets and is part of the "classical" descriptive set – theoretic techniques applied to the case of the κ – reals. In this context, it seems that there is a natural way for generalizing traditional descriptive set theory of Polish spaces to that of "κ – Polish spaces" (that is, "κ – metric spaces" of density κ).

There are many issues related to the κ – reals which we plan to investigate further. For instance, analogues of the *Baire property*, of *Lebesgue measurability*, and the study of the differential and integral calculus of such long reals, just to mention a few. At this point, the possibilities seem open – ended.

3. Future collaboration & Projected publications

Both of the themes that were described above, that is, the *resurrection axioms* and the *long reals*, should be understood as work which is still in progress. We are currently preparing two separate preprints, one for each subject.

There are no concrete plans for me visiting UEA in the next few months. Nevertheless, we will continue to work jointly (alas, from distance) with David Asperó on these topics, presenting our results to colleagues at various seminars and conferences in the near future. Eventually, we expect to submit these two articles (whose exact titles have not been decided yet) for publication in major international journals.

As a final remark of this report, let me take the opportunity to express my gratitude to the ESF for the support which I received and which made this research possible. I have really enjoyed my time working with David Asperó on these, in our opinion,

intriguing and engaging mathematical themes. I am also grateful for the chance to meet the members of the mathematics department of UEA and, in particular, the very nice people of the local logic group, to whom I am trully indebted for their warm hospitality. In total, I believe that I have greatly benefitted from this research visit and I am looking forward to similar visits in the future.

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