# The structure of compact zero-dimensional semilattices (INFTY Exchange Grant No. 3635)

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## Report

During my visit I worked with Professor Robert Bonnet mostly on questions related to semilattice structures on compact spaces. We have also continued our joint cooperation on some properties of Boolean algebras related to generating sets and ultrafilters. We have almost finished two articles [BBK], [BK1] and the third one [BK2] is in preparation. The details are explained below.

### Scattered compact semilattices

One of the main questions we have considered is the existence of continuous semilattice structures on typical small scattered semilattices. It is known [1] that the compact space obtained from a maximal almost disjoint family cannot have a semilattice structure. We have proved:

**Proposition 1.** *Given a scattered compact space $K$ of height 3, there exists a compact semilattice $L \supseteq K$ such that $L \setminus K$ consists of isolated points.*

This simple result actually hides a canonical way of constructing semilattices. More precisely, if $\mathcal{A}$ is an almost disjoint family, let us denote by $K_{\mathcal{A}}$ the 0-dimensional compact space whose Boolean algebra of clopen sets is generated by $\mathcal{A}$ plus all the finite subsets of $\bigcup \mathcal{A}$. Proposition 1 says that, given such a family $\mathcal{A}$ there is another almost disjoint family $\mathcal{A}^\star$ such that $K_{\mathcal{A}^*}$ carries a semilattice structure.

Recall that an almost disjoint family $\mathcal{A}$ on $\mathbb{N}$ is *Lusin* (see [3]) if no subset of $\mathbb{N}$ separates uncountably many elements of $\mathcal{A}$. Let us call an indexed almost disjoint family $\mathcal{A} = \{A_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{P}(\mathbb{N})$ *special* if for every $n \in \mathbb{N}$, for every $\alpha < \omega_1$, the set

$$\{\xi < \alpha \colon A_\xi \cap A_\alpha \subseteq n\}$$

is finite. Actually, the family constructed by Lusin in [3] is special and every special family is Lusin. We have proved:

**Theorem 2.** *There exists a special Lusin family* $\mathcal{A} = \{A_\alpha\}_{\alpha<\omega_1}$ *such that* $K_\mathcal{A}$ *is a semilattice.*

We still do not know whether there is a Lusin family $\mathcal{A}$ such that $K_\mathcal{A}$ has no semilattice structure.

Recall that a *ladder system* on an uncountable regular cardinal $\kappa$ is a family $\mathcal{C} = \{c_\delta\}_{\delta\in S}$ where $S \subseteq \kappa$ is a stationary set, such that each $c_\delta \subseteq \delta$ is order isomorphic to $\omega$ and $\sup c_\delta = \delta$ for every $\delta \in S$. Clearly, every ladder system is an almost disjoint family.

We have:

**Theorem 3.** *For every regular cardinal* $\kappa > \aleph_0$*, for every stationary set* $S \subseteq \kappa$ *consisting of ordinals of cofinality* $\omega$ *there exists a ladder system* $\mathcal{C} = \{c_\delta\}_{\delta\in S}$ *such that the space* $K_\mathcal{C}$ *has a semilattice structure.*

As a corollary, we get the existence of many pairwise non-homeomorphic compact semilattices coming from almost disjoint families on $\omega_1$. Again, we still do not know whether *every* compact space coming from a ladder system admits a semilattice structure.

The results described above will be contained in [BBK], where we also make some observations on possible classification of isomorphic types of semilattice structures on simple spaces, like $\omega + 1$ or the one-point compactification of an uncountable discrete space.

One has to mention that a preliminary version of [BBK] (a joint work with Taras Banakh) existed few years ago, containing the following interesting result, which was one of the main motivations for our study:

**Theorem 4.** *Let* $K$ *be a compactification of* $\mathbb{N}$ *whose remainder is homeomorphic to* $[0, \omega_1]$*. Then* $K$ *has no separately continuous mean. In particular,* $K$ *has no semilattice structure.*

We still do not have any ZFC example of a compact scattered space of countable height and of size $\omega_1$ which has no semilattice structure.

The work [BBK] is almost in the final stage.

## Compact semilattices and the duality

We study a Pontryagin-type duality for semilattices. Our approach is slightly different than that in the book of Hofmann, Mislove & Stralka [2]. Namely, we consider meet semilattices with zero $\mathbb{S} = \langle S, \wedge, 0\rangle$ and the dual object $\mathbb{S}$ is just the space of all homomorphisms into $\{0, 1\}$. In case $\mathbb{S}$ carries a topology, by a *homomorphism* we understand a continuous homomorphism, so $\mathbb{S}^*$ is a closed subspace of the space of continuous functions on $\mathbb{S}$. Starting from a discrete semilattice, the dual object is a compact 0-dimensional semilattice, and vice-versa. The duality says that $\mathbb{S}^{**}$ is canonically isomorphic to $\mathbb{S}$ whenever $\mathbb{S}$ is either discrete or compact (and 0-dimensional).

It turns out that some basic properties of this duality are not specifically stated in the literature, therefore we have decided to write a survey-type article [BK2], including some new observations and results, emphasizing on topological aspects of the compact semilattice theory. In particular, using the duality, we show the existence of compact 0-dimensional semilattices which have a continuous homomorphism onto every compact 0-dimensional semilattices of a prescribed weight $\kappa \geqslant \aleph_0$. More precisely, we have:

**Theorem 5.** *Assume $\kappa^{<\kappa} = \kappa$. Then there exists a unique up to isomorphism compact 0-dimensional semilattice $\mathbb{S}_\kappa$ of weight $\kappa$ and with the following properties:*

(1) *Every compact semilattice of weight $\leqslant \kappa$ is a homomorphic image of $\mathbb{S}$.*

(2) *Given a compact semilattice $K$ of weight $< \kappa$, given epimorphisms $f_0, f_1 \colon \mathbb{S} \to K$, there exists an isomorphism $h \colon \mathbb{S} \to \mathbb{S}$ such that $f_1 = f_0 \circ h$.*

In particular, assuming the Continuum Hypothesis, there exists a compact 0-dimensional semilattice $\mathbb{S}_{\aleph_1}$ with properties (1) and (2). We believe that this compact space does not have a compatible meet semilattice structure with the unit (which would answer our question stated in the proposal).

The existence of a semilattice satisfying (1) and (2) can be proved in ZFC, although its weight may be strictly bigger than $\kappa$ and it may not be unique.

The work [BK2] is in preparation now; we expect to have a complete version by the end of 2012.

## Corson-like Boolean algebra

We have improved and extended some results from the paper [BK1], originated during his INFTY short exchange visit in Prague (May 2012). One of the new results is:

**Theorem 6.** *Let $\mathbb{B}$ be an interval Boolean algebra and let $\kappa$ be an uncountable regular cardinal such that $|\mathbb{B}| > \kappa$. Then for every generating set $G \subseteq \mathbb{B}$, there exists $a \in \mathbb{B}^+$ such that $|G \cap p| \geqslant \kappa$ whenever $a \in p \in \operatorname{Ult}(\mathbb{B})$.*

We have also studied compact 0-dimensional semilattices in the context of Corson-like properties, obtaining a variant of Theorem 6 for a wide class of semilattices, including all distributive lattices.

We are currently consulting our results with Professor Stevo Todorcevic, whose remarks already helped us to improve some arguments and the presentation. The work [BK1] is almost in the final stage.

# References

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[1] T. Banakh, O. Gutik, M. Rajagopalan, *On algebraic structures on scattered compacta*, Topology Appl. **153** (2005) 710–723

[2] K.H. Hofmann, M. Mislove, A. Stralka, *The Pontryagin duality of compact 0-dimensional semilattices and its applications*. Lecture Notes in Mathematics, Vol. 396. Springer-Verlag, Berlin-New York, 1974

[3] N.N. Luzin, *On subsets of the series of natural numbers*. (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. 11 (1947) 403–410