Report on the workshop:

Contact topology in higher dimensions

Organizers:	John Etnyre, Emmanuel Giroux, and Klaus Niederkrüger
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1. Summary

The workshop *Contact topology in higher dimensions* was held at the American Institute of Mathematics (Palo Alto, California) from May 21 to May 25, 2012. It was sponsored by AIM, the NSF and the ESF network CAST.

The motivation for organizing this workshop was the evidence that, in the field of symplectic geometry which has been expanding extremely fast in the last three decades, high dimensional contact geometry remains relatively underdeveloped. The goal of the workshop was to stimulate new research in this area by discussing recent progress in the domain and pointing out important open problems for which it seems we currently have enough tools to make significant advances.

Following the AIM guidelines, our workshop was not centered on speakers presenting their recent research results, but instead the aim was to point out the most important problems in the field and discuss possible approaches to tackle them. In the mornings several experts gave survey talks explaining the current state of the art regarding certain questions, and in the afternoon the participants would break into small groups to work on specific problems.

2. Description of the scientific content of and discussions at the event

From Monday to Thursday, the two morning lectures focused each day on a specific theme:

- contact structures on Monday — with a survey by Giroux on the main methods we know to construct contact structures and then a survey lecture by Massot on the fillability properties which relate contact geometry to symplectic geometry;
- Legendrian submanifolds on Tuesday — with an expository lecture by Traynor on Legendrian knots and their invariant, and then a talk by Murphy on the class of flexible Legendrian knots she recently discovered;

- Stein/Weinstein manifolds on Wednesday — with a survey lecture by Eliashberg emphasizing the existence of great flexibility in Stein/Weinstein structures and then a talk by Abouzaid describing recent constructions of exotic Stein manifolds due to McLean and also to Seidel and himself;
- contactomorphisms on Thursday — with a survey talk by Sandon on squeezing problems and their relations with the geometry of contactomorphism groups, and then a talk by Fraser describing a Sandon-type metric she recently obtained with Polterovich and Rosen on some contactomorphism groups.

The Friday morning lectures were devoted to a more detailed description of two important recent results: the general construction of contact structures in dimension 5 — a talk by Presas on his joint work with Casals and Pancholi — and the construction of Legendrian submanifolds with many non-trivial invariants — a talk by Bourgeois on his joint work with Sabloff and Traynor.

On Monday, Tuesday and Friday, the afternoon started with open problem sessions in which many questions and ideas were discussed (see the notes taken by Courte during these sessions). For the remainder of these afternoons, as well as the afternoon sessions Wednesday and Thursday, the participants broke into several working groups to study specific problems or topics.

More specifically, the groups were based around the following topics:

(1) How can one distinguish the contact structures on $S^3 \times S^2$ associated with the open books with page T^*S^2 and monodromy a positive even power of the right-handed Dehn twist?
(2) How can one compare the various discrete biinvariant metrics on contactomorphism groups which have been defined so far (by Sandon, Fraser-Polterovich-Rosen and Colin-Sandon)?
(3) How can we define and compute Lagrangian cobordism invariants for Legendrian knots?
(4) Is there any nontrivial quasimorphism on contactomorphism groups which might lead to a Milnor-Wood-type inequality for contact bundles over surfaces?
(5) If the complement of a Legendrian submanifold contains a generalized overtwisted disk (such as a "plastikstufe" or a "bLob"), is the submanifold necessarily loose?
(6) What could be a useful extension of the convex surface theory in high dimension?

The groups made the following progress.

Group 1. This group thought about how to distinguish the contact structures on the open books arising from different iterations of the Dehn-Seidel twist on T^*S^n, especially with $n = 2$. Odd iterations lead to structures that can be distinguished (for example using contact homology, see Ustilovsky

and van Koert), but so far known methods fail for even powers. The members of this group tried in particular to prove the following conjecture: if the contact manifold for the k-th iterated twist is denoted by M_k ($k \geq 1$), then there exists no exact symplectic cobordism from M_k (concave) to M_i (convex) when $i < k$. The strategy for proving this was loosely based on the work that Latschev and Wendl had done on algebraic torsion in SFT: one can define a "U-map" (analogous to the U-map in ECH) that counts index 2 holomorphic curves through a generic point in the symplectization, and then define a numerical contact invariant by asking "what is the smallest integer $k \geq 0$ such that $\hbar^k$ lies in the image of U?" The group hoped to understand the holomorphic curves in M_k well enough to show that this numerical invariant for M_k equals $k-1$; ideally this argument could also be translated into a direct holomorphic curve argument to show the nonexistence of the cobordism without relying on the formalism of SFT. While this program was not completed during the workshop several of the members left with a feeling of cautious optimism.

The group also thought a little bit about whether $\mathbb{R}P^{2n-1}$ is exactly fillable for $n > 2$. A partial "result" obtained from these discussions was that if it has an exact filling, the filling must be simply connected. This is because, roughly speaking, a non-simply-connected filling would have a universal cover which is a geometrically bounded exact filling of potentially multiple copies of the projective space and/or the sphere, and multiple copies can be ruled out using some combination of the Eliashberg-Floer-McDuff theorem (on fillings of the sphere) with closely related holomorphic curve arguments for the projective space.

Group 2. The group started by comparing three different types of biinvariant metrics that have been defined recently on certain contactomorphism groups, namely: the ones based on spectral invariants linked to generating functions (on $\mathbb{R}^{2n} \times S^1$ by Sandon, and its extension to $T^*X \times S^1$ by Zapolski); the one based on the notion of discriminant and translated points (by Colin-Sandon for all contact manifolds); and the one using a partial order on the universal cover of the contactomorphism group (by Fraser-Polterovich-Rosen for certain manifolds with 1-periodic Reeb flow).

This meeting provided an excellent opportunity of exchange on this topic, because the latter two metrics are still work in progress. One of the main questions discussed was if these metrics might be equivalent when they are defined on the same manifold. Unfortunately no conclusion was reached on this issue.

Another set of questions centered on the possible relations of these metrics with orderability and existence of quasimorphisms on the contactomorphism group, and contact non-squeezing.

Group 3. This group discussed various techniques to construct Lagrangian cobordisms and specifically concentrated on Lagrangian caps (cobordisms to

the empty set) and fillings (cobordisms from the empty set). They also considered the construction of Legendrian submanifolds in contact manifolds. In certain settings in high dimension, the h-principle implies that such Lagrangian caps exist, but currently there are virtually no known concrete constructions.

They began by considering the three dimensional case – here Murphy discussed her construction of a Lagrangian cap for a twice-stabilized unknot, and the group used this to derive a new result, that any sufficiently stabilized Legendrian knot has a Lagrangian cap. This leads to a possibly interesting direction for future research – for a given topological knot type, what is the largest Thurston-Bennequin number for a Legendrian knot in that class that has a Lagrangian cap?

The group also worked for a while on constructing other Lagrangian caps. The models for Lagrangian caps of 1-dimensional Legendrian knots give rise to caps for some higher-dimensional Legendrians as well, but a lot more work is needed.

Group 4. The recent work of Colin and Sandon suggests that there exists a map (whose definition is related to their discriminant metric) that could provide a quasimorphism for certain contactomorphism groups. In particular, the members of this group checked that this map coincides with the Milnor-Wood quasimorphism on the diffeo/contacto-morphism group of the circle. Though, they were unable to prove that this is a quasimorphism in some higher dimensional examples (except for the real projective space, in which case the result is due to Givental), the discussions clarified several interesting and important points.

Group 5. This group discussed if every Legendrian submanifold L that lies in the complement of a plastikstufe is loose in the sense of Murphy. For certain types of plastikstufes, it seems very likely that this is indeed true: one can isotope a part of the knot into a position parallel to the core of the plastikstufe, and then reduce the question to destabilizing slice-wise using the analogous 3-dimensional result.

Moreover, it was observed that some distinct contact manifolds that arise as the boundary of exotic Stein manifolds (due to McLean and Abouzaid-Seidel) become contactomorphic after applying a "generalized Lutz twist" or connect summing with a fixed PS-overtwisted contact structure on the sphere. Both these observations point to plastikstufe implying a certain flexibility of contact structures, lending credence to the possibility that PS-overtwisted might be a good generalization of the notion of overtwistedness in dimension three.

After these positive results, the discussion centered on understanding if the looseness property could also be proved for knots lying in the complement of more general objects than plastikstufes, as for example for bLobs or for sufficiently "large" neighborhoods of the overtwisted disk.

Group 6. This group studied two specific questions. The first concerns the existence of convex hypersurface in high dimensions and the second concerned the notion of a bypass and overtwisted disk in high dimensions. There were several ideas discussed concerning the first question, but at the moment it seems quite difficult to construct convex hypersurfaces in general. Honda discussed a specific model for a bypass in all dimensions that could then be doubled to give a conjectural overtwisted disk. There was some discussion about the existence of such things in the presence of plastikstufe and bLobs, but no specific results were obtained.

3. Assessment of the results and impact of the event on the future directions of the field

The future directions resulting from the discussions during the workshop have already been summarized in detail in the previous section. Furthermore, a list of the most important questions formulated during the workshop can be downloaded on the website of the American Institute of Mathematics (`http://www.aimath.org/pastworkshops/contacttop.html`).

As conclusion, we only add that all in all, it is clear the workshop has stimulated a great deal of work and new collaborations. In particular, it has managed to attract interest of several researchers that had previously mostly worked on 3-dimensional questions, and we believe that this will spawn active research in higher dimensional contact topology.

Appendix A. Programme of the meeting and full list of speakers and participants.

Contact topology in higher dimensions
Workshop Schedule

Monday, May 21, 2012		**Tuesday, May 22, 2012**	
9:00 am	Welcome & Introductions Emmanuel Giroux Patrick Massot	9:00 am	Lisa Traynor Emmy Murphy
2:00 pm	Discussion	2:00 pm	Discussion Working Groups
	Happy Hour		Happy Hour
		6:45 pm	Workshop Banquet

Wednesday, May 23, 2012		**Thursday, May 24, 2012**	
9:00 am	Yasha Eliashberg	9:00 am	Sheila Sandon
	Mohammed Abouzaid		Maia Fraser
2:00 pm	Working Groups	2:00 pm	Group Reports
			Working Groups
	Happy Hour		Happy Hour

Friday, May 25, 2012	
9:00 am	Group Reports
	Working Groups
2:00 pm	Working Groups
	Happy Hour

Participants.

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