Scientific Report
Workshop on Symplectic Field Theory VI

Time: July 21–27, 2012
Place: Ludwig-Maximilians-Universität München
Organizers: P. Albers, K. Cieliebak, J. Latschev, K. Mohnke, M. Schwarz

Summary

The series of Workshops on Symplectic Field Theory started in 2005 and has been highly successful in making some of the important recent developments in holomorphic curve theories accessible to a wide range of mathematicians including graduate students, postdocs, and some established researchers.

The format consists in having an established expert give a series of lectures on the main topic chosen, accompanied by several shorter lectures series and discussion sessions which present additional or complementary material. The main speaker for the 2012 event was Michael Hutchings (Berkeley), with additional lectures given by Dusa McDuff (Barnard College, New York), Chris Wendl (University College London), Yi-Jen Lee (Purdue), and Paolo Ghiggini (Nantes), and discussion sessions held by Keon Choi (Berkeley) and Vinicius Gripp (Berkeley).

The topic of this workshop was Embedded Contact Homology (ECH). This is a holomorphic theory in dimension four which has been developed over the past ten years by Michael Hutchings and Clifford Taubes. It has close relations to Seiberg-Witten theory and Heegaard-Floer theory. The spectacular applications of ECH include the proof and various refinements of the Weinstein conjecture and Arnold's chord conjecture in dimension three, and the solution of the symplectic embedding problem for ellipsoids in dimension four. It was the goal of this workshop to explain the foundations of ECH as well as some of its applications.

The workshop had 66 registered participants, mostly graduate students and young postdocs.

Description of the scientific content of and discussions at the event

Precourses

As has become customary for the Workshops on Symplectic Field Theory, the main event was preceded by 3 precourses on the weekend before the workshop

(July 21–22) to recall useful background. The titles and content of the precourses were:

Contact Topology in Dimension 3 (Patrick Massot)

Examples and definition of contact manifolds. Symplectic fillings and cobordisms (strong, weak, exact). The overtwisted/tight dichotomy; statement that fillability implies tightness. Contact surgery gives an exact cobordism; conjecture that surgery preserves tightness.

Holomorphic Curves in Dimension 4 (Sam Lisi)

LECTURE 1. Basic properties of holomorphic curves.

Moduli space of unparametrized closed holomorphic curves. The index formula. Transversality for generic J (statement and sketch of proof). Gromov compactness (statement and a detailed example).

LECTURE 2. Holomorphic curves in dimension 4.

Automatic transversality (with proof). Positivity of intersection (heuristic argument). Adjunction formula (sketch in case of integrable J).

LECTURE 3. Punctured holomorphic curves.

Complex structures on symplectizations. Cobordisms with cylindrical ends. Hofer energy. A punctured finite energy curve converges to Reeb orbits at punctures. Conley-Zehnder index illustrated by examples in $S^1 \times D^2$. Some remarks about spectral flow. Fredholm index formula for holomorphic curves in symplectizations (statement and an attempt at a linear example where we can see by hand what is going on). Statement of Dragnev's transversality theorem. Cartoon picture of SFT compactness, with an example of Morse breaking.

Seiberg-Witten Theory (Tim Perutz)

LECTURE 1. The SW equations in 2 and 4 dimensions.

The 2D case: The vortex equations as a warm-up case; their "Taubes parameter" τ. The canonical (empty) vortex. Global description of the moduli space. Localization of solutions when $\tau >> 0$. The SW equations in 4D. (Ir)reducible monopoles. Using a symplectic form to perturb the equations; the canonical monopole and Taubes's constraints; brief mention of the Kahler case.

LECTURE 2. Holomorphic curves and Reeb orbits versus SW monopoles.

The SW invariants (briefly). Statement of SW=Gr; existence of canonical curves via Taubes duality. The Taubes limit and localization along holomorphic curves. Monopoles approximated by holomorphic sections of the vortex bundle along an embedded holomorphic curve. The Weinstein conjecture: cheap reducible solutions persist under perturbation (Floer homology) so one can pass to the Taubes limit; difficulty of controlling the length of the approximate Reeb orbits.

Lecture series

The main programme (July 23-27) consisted of 5 lecture series, complemented by informal discussion sessions. The titles and content of the lecture series were:

Embedded Contact Homology (Michael Hutchings)

LECTURE 1. Origins of PFH and ECH from Seiberg-Witten theory.
Taubes's "Seiberg-Witten = Gromov" theorem asserts that the Seiberg-Witten invariants of a closed symplectic four-manifold are equal to Taubes's "Gromov invariant", which is a certain count of mostly embedded holomorphic curves. At the beginning of this story, we wanted to define an analogue of Taubes's Gromov invariant for three-dimensional contact manifolds (and mapping tori) which would agree with Seiberg-Witten Floer homology of the three-manifold. This leads naturally, with a few tricky points, to the definition of embedded contact homology (and periodic Floer homology). Taubes has subsequently shown that ECH indeed agrees with Seiberg-Witten Floer homology, and this fact is essential for most of the applications of ECH.

LECTURE 2. The ECH index inequality and the definition of ECH.
We will explain the ECH index inequality, which is the main nontrivial part of the definition of ECH. It tells us which holomorphic curves to count in the differential, and why such a count is well-defined.

LECTURE 3. The U map and applications to improvements of the Weinstein conjecture.
ECH has various additional structures on it; one of the most important of these is the U map. The existence of holomorphic curves counted by the U map is a key ingredient in several applications. One application is to slightly improve on the Weinstein conjecture, for example by proving that there is more than one Reeb orbit, or by extending the class of vector fields for which a closed orbit is known to exist.

LECTURE 4. Cobordism maps and applications to the chord conjecture and symplectic embedding problems.
Exact symplectic cobordisms between contact three-manifolds induce maps on ECH which respect a symplectic action filtration. This fact can be used to prove that every Legendrian knot in a closed contact three-manifold has a Reeb chord. It can also be used to define "ECH capacities", which give quantitative obstructions to symplectically embedding one symplectic four-manifold (usually with boundary) into another.

LECTURE 5. Calculation of the ECH of T^3 and some related ECH capacities.
To the extent that time permits, we will explain the detailed calculation of the ECH of the three-torus with certain contact forms. This calculation allows one to compute many examples of ECH capacities. For example it is used in the calculation of the ECH capacities of polydisks.

Symplectic Embeddings (Dusa McDuff)

LECTURE 1. Symplectic embeddings in dimension 4 and above.
Embeddings of ellipsoids into ellipsoids in dimension 4 and above, Guth's counterexample to Hofer's conjecture, embeddings of balls into tori and other 4-manifolds, open problems.

LECTURE 2. Methods for constructing symplectic embeddings.
Using inflation in rational and ruled 4-manifolds to embed ellipsoids, Guth's method, other direct methods.

LECTURE 3. Calculation of the McDuff–Schlenk capacity function for 4-dimensional ellipsoids.
Resolving toric singularities, continued fractions, obstructions from Fibonacci numbers, sample calculations.

Holomorphic Curves in Dimension 4 and the ECH Contact Invariant (Chris Wendl)

LECTURE 1. Punctured holomorphic curves in dimension 4.
Asymptotic formulas, automatic transversality, intersection theory, finite energy foliations.

LECTURE 2. Planarity and symplectic filling obstructions.
The ECH contact invariant and vanishing results, application to fillability, the U-map and planarity obstructions.

LECTURE 3. Measuring "degrees of tightness".
A brief digression on SFT and algebraic torsion, Hutchings's filtration via J_+, obstructions to exact symplectic cobordisms

Embedded contact homology and Seiberg-Witten Floer homology (Yi-Jen Lee)

LECTURE 1. Counting curves and counting Seiberg-Witten solutions: outline and key ideas of Taubes's proof of "Gromov = Seiberg-Witten", and its Floer-theoretic extensions.

LECTURE 2. Filtered variants of Seiberg-Witten Floer homology and ECH, application to a proof of "Seiberg-Witten Floer homology = Heegaard Floer homology".

Embedded contact homology and Heegaard Floer homology (Paolo Ghiggini)

LECTURE 1. Definition of Heegaard Floer homology. Statement of the isomorphism. Open book decompositions. The map from HF^+ to ECH.

LECTURE 2. The hat version of ECH. The map from $\widehat{HF}$ to $\widehat{ECH}$. The map from $\widehat{ECH}$ to $\widehat{HF}$. Final considerations.

Discussion sessions

In the first two discussion sessions, Keon Choi and Vinicius Gripp presented some detailed computations of ECH, in particular for the example of an ellipsoid. In the third discussion session, Michael Hutchings explained details of the gluing of embedded holomorphic curves and the proof that $\partial^2 = 0$.

At the end of the event there was an open problem session during which many interesting open problems were suggested by the following participants: Patrick Massot, Michael Hutchings, Chris Wendl, Felix Schlenk, Emmanuel Opshtein, Fabian Ziltener, and Frol Zapolsky.

Assessment of the results and impact of the event on the future directions of the field

The Workshops on Symplectic Field Theory serve two goals: introducing researchers in the area of Contact and Symplectic Topology to an exciting new subject, and providing an opportunity for young researchers to meet other people in the field.

Based on the feedback from participants, the first goal was overwhelming achieved. This was largely due to the excellent and well-prepared speakers. Another contributing factor was that, compared to previous SFT workshops, the topic of this one was more focused and thus easier to cover within a week.

Long coffee and lunch breaks provided ample time for informal discussions, which often continued after the lectures in the beer gardens.

The global assessment of this workshop is very positive. Indeed, motivated by this success and the overwhelming feedfack from many participants, preliminary discussions for a seventh Workshop on Symplectic Field Theory are already taking place.

Programme of the meeting

Saturday
9.00 - 10.30 P. Massot
11.00 - 12.30 S. Lisi
14.30 - 16.00 S. Lisi

Sunday
9.00 - 10.30 S. Lisi
11.00 - 12.30 T. Perutz
14.30 - 16.00 T. Perutz

Monday
9.00 - 10.30 M. Hutchings
11.00 - 12.30 D. McDuff

14.30 - 16.00 C. Wendl
16.30 - 17.30 Discussion Session (K. Choi)

Tuesday
9.00 - 10.30 M. Hutchings
11.00 - 12.30 D. McDuff

14.00 - 15.00 Y.-J. Lee
15.00 - 16.00 Discussion Session (V. Gripp)
16.30 - 17.30 P. Ghiggini

Wednesday
9.00 - 10.30 M. Hutchings
11.00 - 12.30 D. McDuff

Thursday
9.00 - 10.30 M. Hutchings
11.00 - 12.30 C. Wendl

14.00 - 15.00 Y.-J. Lee
15.00 - 16.00 Discussion Session (M. Hutchings)
16.30 - 17.30 P. Ghiggini

18.00 Buffet

Friday
9.00 - 10.30 M. Hutchings
11.00 - 12.30 C. Wendl

14.30 - 16.00 Open Problem Session

List of speakers and participants

Alves, Marcelo
Anjos, Silvia
Avdek, Russell
Bauer, David
Blaga, Adara
Borman, Matthew Strom
Bottman, Nate
Bounya, Cedric
Bowden, Jonathan
Casals, Roger
Chantraine, Baptiste
Cieliebak, Kai
Cioba, Alexandru
Choi, Keon
Espina, Jacqui
Fanoe, Andrew
Farris, David
Fauck, Alexander
Frenkel, David
Fromm, Viktor
Fuchs, Urs
Futaki, Masahiro
Furukawa, Ryo
Gadbled, Agnes
Gerstenberger, Andreas
Ghiggini, Paolo
Golovko, Roman
Gripp, Vinicius
Gutt, Jean
Gyenge, Adam
Hamilton, Mark
Horvath, Ramon
Huster, Johannes
Hutchings, Michael
Karlsson, Cecilia
Kasuya, Naohiko
Kirchner, Fabian
Latschev, Janko
Lee, Yi-Jen
Lisi, Samuel
Maiti, Arun
Massot, Patrick
McDuff, Dusa
Opshtein, Emmanuel
Ott, Andreas
Perutz, Tim
Rezazadegan, Reza
Rieser, Antonio
Ritter, Alexander
Rot, Thomas
Saglam, Murat
Savelyev, Yasha
Schlenk, Felix
Shelukhin, Egor
Siefring, Richard
Spano, Gilberto
Starostka, Maciej
Stiller, Michael
Uebele, Peter
Vaugon, Anne
Volkov, Evgenij
Wendl, Chris
Xu, Yongzhong
Yoshiyasu, Toru
Ziltener, Fabian
Zehmisch, Kai