Scientific Report

The purpose of this visit was twofold:

(i) Discuss with researchers from the symplectic geometry group of LMU, in particular Kai Cieliebak, possible applications of recent work on Lagrangian intersections [2] and non-intersections [3] in toric manifolds.

(ii) Work with Paul Norbury (University of Melbourne, visiting LMU) on a possible relation between the Kähler geometry of the flag manifold $F(1,2)$, the twistor space of $\mathbf{P}^2$, and the Kähler-Einstein metric on $\tilde{\mathbf{P}}^2$, the 3-point blow-up of $\mathbf{P}^2$, that was recently suggested to me by Michael Atiyah.

Regarding (i), I gave a talk in the Topics in Symplectic Geometry seminar that Kai Cieliebak runs at LMU and discussed with him the possible application of Givental's gluing construction of Lagrangian submanifolds (see e.g. [5]) in the context of toric manifolds. This possibility had already been suggested to Agnès Gadbled by Ivan Smith. Agnès and I have a joint project that aims at giving a moment polytope combinatorial understanding of certain exotic monotone Lagrangian tori in $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ (cf. [6]) and Ivan noticed that our first attempts seem to be related with Givental's construction.

Regarding (ii), Paul and I explored the geometric implications of the embeddings of $F(1,2)$ and $\tilde{\mathbf{P}}^2$ in $\mathbf{P}^2 \times \mathbf{P}^2$ as

$$F(1,2) = \left\{([z_0 : z_1 : z_2], [w_0 : w_1 : w_2]) \in \mathbf{P}^2 \times \mathbf{P}^2 \,,:\ z_0w_0 + z_1w_1 + z_2w_2 = 0\right\}$$

and

$$\tilde{\mathbf{P}}^2 = \left\{([z_0 : z_1 : z_2], [w_0 : w_1 : w_2]) \in \mathbf{P}^2 \times \mathbf{P}^2 \,,:\ z_0w_0 = z_1w_1 = z_2w_2\right\} \,.$$

One can use this set-up to understand certain complex and symplectic relations between these two spaces, such as:

- the relation between the natural Hamiltonian 2-torus actions on these spaces, their moment maps and moment polytopes;
- the fact that the natural anti-holomorphic involution that $F(1,2)$ has as a twistor space induces an anti-holomorphic (and anti-symplectic) involution on $\tilde{\mathbf{P}}^2$ with fixed point set the unique monotone Lagrangian orbit of the 2-torus action.

We noticed that the quotient of $\tilde{\mathbf{P}}^2$ is $\overline{\mathbf{P}}^2$, i..e. $\mathbf{P}^2$ with its opposite orientation, which means that the Kähler-Einstein metric on $\tilde{\mathbf{P}}^2$ is the pull-back by the quotient map of an Einstein metric on $\overline{\mathbf{P}}^2$ with a normal cone singularity along the Clifford torus.

There are some interesting explicit constructions of Einstein metrics on $\mathbf{P}^2$ with normal cone singularities along both complex submanifolds (see for example [1]) and Lagrangian submanifolds (the Atiyah-Hitchin metric and its 1-parameter family of deformations [4]), which gives us some hope of being able to find some explicit description of this Einstein metric on $\overline{\mathbf{P}}^2$ with a normal cone singularity along the Clifford torus (a Lagrangian submanifold).

References

[1] M. Abreu, *Kähler metrics on toric orbifolds*, J. Differential Geom. **58** (2001), 151–187.

[2] M. Abreu and L. Macarini, *Remarks on Lagrangian intersections in toric manifolds*, arXiv:1105.0640, to appear in Transactions of the AMS.

[3] M. Abreu, M. S. Borman and D. McDuff, *Displacing Lagrangian toric fibers by extended probes*, in preparation.

[4] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, 1988.

[5] M. Audin, F. Lalonde and L. Polterovich, *Symplectic rigidity: Lagrangian submanifolds*, in "Holomorphic curves in symplectic geometry" (edited by Michèle Audin and Jacques Lafontaine), Progress in Mathematics **117**, Birkhäuser (1994), 271–321.

[6] A. Gadbled, *On exotic monotone Lagrangian tori in* $\mathbb{C}P^2$ *and* $S^2 \times S^2$, arXiv:1103.3487, to appear in Journal of Symplectic Geometry.