CAST TRAVEL GRANT SCIENTIFIC REPORT: RELATIVE HOFER-ZEHNDER CAPACITIES AND AN ENERGY-CAPACITY INEQUALITY

ANTONIO RIESER

The following is the scientific report for the CAST travel grant for my visit to work with Samuel Lisi at the Université Libre de Bruxelles from August 12th - August 19th, 2012.

1. Purpose of the visit

Our aim was to complete the construction of a Hofer-Zehnder-type capacity $c_{0,L}$ which is defined for a pair (M, L), where (M, ω) is a symplectic manifold and $L \hookrightarrow M$ is a Lagrangian submanifold which is either closed or whose boundary is contained in the boundary of M, i.e. $\partial L \subset \partial M$. Our motivation for this is that, first, such a capacity would be an upper bound for the relative Gromov width $Gr_{\mathbb{R}}(M, L)$ of the Lagrangian submanifold L, where $Gr_{\mathbb{R}}(M, L)$ is defined as the supremum over r over all symplectic embeddings $\{\phi : B(r) \to M\}$ with $\phi^{-1}(L) = B_{\mathbb{R}}(r)$, $B_{\mathbb{R}}(r) := B(r) \cap \mathbb{R}^n$. Second, we conjecture that, for closed Lagrangians in $\mathbb{R}^{2n}$, $c_{0,L}(\mathbb{R}^{2n}, L) \leq E(L)$, where $E(L)$ is the displacement energy of the Lagrangian. If both of these conjectures are true, then this would prove, in particular, that the Gromov width of all Lagrangian tori of $\mathbb{R}^4$ is finite, resolving a conjecture of Biran and Cornea.

2. Work carried out during the visit, main results obtained

During this visit, we concentrated on using variational methods to prove the existence of $c_{0,L}$, the most difficult part of which was proving that

$c_{0,L}(B(1), B_{\mathbb{R}}(1)) < \pi$. To descibe our result, we make the following definition.

Definition 2.1. Let (M, ω) be a symplectic manifold, and let $L \hookrightarrow M$ be a Lagrangian submanifold which is either closed or such that $\partial L \subset \partial M$. A *relative symplectic capacity* c_L is a map $(M, L, \omega) \mapsto c_0(M, L, \omega) \in [0, \infty)$ such that

(1) If there exists a relative symplectic embedding $\phi : (N, K, \omega_1) \to (M, L, \omega_2)$, then $c(N, K, \omega_1) \leq c(M, L, \omega_2)$.
(2) $c(M, L, \alpha\omega) = |\alpha| c(M, L, \omega), \alpha \in \mathbb{R}$
(3) $c(B(1), B_{\mathbb{R}}(1), \omega_0) = \pi$

Our main result obtained during this visit was the following:

Theorem 2.2. *Let M be a symplectic manifold and let $L \hookrightarrow M$ be a Lagrangian submanifold. Define the set $\mathcal{H}_a(M, L)$ to be the set of real functions $H : M \to \mathbb{R}$ which satisfy:*

(1) *There exists a compact set $K \subset M$ (depending on H) such that $K \subset M \backslash \partial M$, $\emptyset \neq K \cap L \subsetneq L$, and*

$$H(M \backslash K) = m(H) \text{ (a constant).}$$

(2) *There exists an open set $U \subset M$ (depending on H), with $\emptyset \neq U \cap L \subsetneq L$, and on which $H(U) \equiv 0$.*
(3) *$0 \leq H(x) \leq m(H)$ for all $x \in M$.*
(4) *$\forall x \in L, X_H(x) \neq 0 \implies X_H \notin TL$, and for any $t_0 > 0$, all the solutions of $\dot{x} = X_H(x), \{\phi_H(x, 0), \phi_H(x, t_0)\} \in L$, are either such that $x(t)$ is constant for all $t \in \mathbb{R}$, or such that $t_0 > 1/2$.*

Let $\|H\|$ denote the Hofer norm of a function $H : M \to \mathbb{R}$. The number

$$c_{0,L}(M, L) := sup\{\|H\| | H \in \mathcal{H}_a(M, L)\}$$

is a relative symplectic capacity.

In the above, X_H denotes the Hamiltonian vector field associated with H, and $\phi_H : M \to M$ is the induced Hamiltonian diffeomorphism.

It also follows that

Proposition 2.3. $Gr_{\mathbb{R}}(M, L) \leq c_{0,L}(M, L)$.

3. Future collaboration with host institution

Our conjecture about the energy-capacity inequality for $c_{0,L}$ remains open, and we are currently working to resolve it. During our visit, however, it became apparent that it would be extremely difficult to use variational methods to prove this inequality, which has prompted us to investigate the relationship between the relative Hofer-Zehnder capacity $c_{0,L}$ and the action spectrum in various types of Floer and infinite dimensional Morse theories, when these can be defined. Dr. Lisi is moving to the Université de Nantes beginning in the fall of 2012, so future collaboration on this project will be between the Technion and the Université de Nantes.

4. Projected publications resulting from this grant

We anticipate that one article with the details of our construction and the proof of the above theorem will directly result from this visit.