CAST TRAVEL GRANT SCIENTIFIC REPORT: LAGRANGIAN NON-SQUEEZING AND A LAGRANGIAN ENERGY-CAPACITY INEQUALITY

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The following is the scientific report for the CAST travel grant for my visit to work with Samuel Lisi at the Université de Nantes from Feruary 10th to February 25th, 2013.

1. Purpose of the visit

During this visit, our intent was to study several issues regarding the Hofer-Zehnder type capacities $c_1(M, L)$ that we constructed earlier. These capacities are defined for a pair (M, L), where (M, ω) is a symplectic manifold and $L \hookrightarrow M$ is a properly embedded Lagrangian submanifold. Our original motivation for this construction was that if, for closed Lagrangians in $\mathbb{R}^{2n}$, one can show that $c_1(\mathbb{R}^{2n}, L) \leq E(L)$, where $E(L)$ is the displacement energy of the Lagrangian, then this would imply that the Gromov width of any Lagrangian tori in $\mathbb{R}^4$ is finite, resolving a conjecture of Biran and Cornea. The inequality above is the analogue in our case to the energy-capacity inequality for the Hofer-Zehnder capacity c_0. Our specific aim for this visit was threefold. First, we wanted to extend the construction of our capacity to co-isotropic submanifolds. Second, we wanted to establish some basic cases of the energy-capacity inequality, lending support for our conjecture that it is true in all cases. Third, we wanted to derive relative embedding obstructions using our capacity.

2. Work carried out during the visit, main results obtained

We made progress on all three goals during this visit. First, we checked the main points necessary to adapt our construction to coisotropic submanifolds. As in the Lagrangian case, we make the following two definitions.

Definition 2.1. Let (M, ω) be a symplectic manifold, and let $N \hookrightarrow M$ be a properly embedded coisotropic submanifold. A *relative symplectic capacity* c_1 is a map $(M, N, \omega) \mapsto c_1(M, L, \omega) \in [0, \infty)$ such that

(1) If there exists a relative symplectic embedding $\phi : (M_1, N_1, \omega_1) \to (M_2, N_2, \omega_2)$, then $c(M_1, N_1, \omega_1) \leq c(M_2, N_2, \omega_2)$.
(2) $c(M, N, \alpha\omega) = |\alpha| c(M, N, \omega), \alpha \in \mathbb{R}$
(3) $c(B(1), B_{n+k}(1), \omega_0) = \pi$, where $B_{n+k}(1) := B(1) \cap \mathbb{R}^{n+k} \subset \mathbb{R}^{2n}$.

Definition 2.2. Let M be a symplectic manifold and let $N \hookrightarrow M$ be a properly embedded coisotropic submanifold. Define the set $\mathcal{H}_a(M, N)$ to be the set of real functions $H : M \to \mathbb{R}$ which satisfy:

(1) There exists a compact set $K \subset M$ (depending on H) such that $K \subset M \backslash \partial M$, $\emptyset \neq K \cap N \subsetneq N$, and

$$H(M \backslash N) = m(H) \text{ (a constant).}$$

(2) There exists an open set $U \subset M$ (depending on H), with $\emptyset \neq U \cap N \subsetneq N$, and on which $H(U) \equiv 0$.
(3) $0 \leq H(x) \leq m(H)$ for all $x \in M$.
(4) The solutions of $\dot{x} = X_H(x), x(0) \in N$, where X_H denotes the Hamiltonian vector field associated with H, are either such that $x(t)$ is constant for all $t \in \mathbb{R}$, or such that $\{t | x(t) \cap N \neq \emptyset\} \subset \{0\} \cup (1/2, \infty)$.

Our theorem then becomes

Theorem 2.3. *Let M be a symplectic manifold and let $N \hookrightarrow M$ be a properly embedded coisotropic submanifold. Let $\|H\|$ denote the Hofer norm of a Hamiltonian $H : M \to \mathbb{R}$. The number*

$$c_1(M, N) := sup\{\|H\| | H \in \mathcal{H}_a(M, N)\}$$

is a relative symplectic capacity.

For our relative embedding obstructions for the pairs $(B(1, p), B_{n+k}(1))$ and $(Z(1, p), Z^{n+k}(1))$, where $B(1, p)$ and $Z(1, p)$ are the unit ball and symplectic cylinder, respectiely, of radius centered at point $(p, 0) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2}, |p| < 1$, our main advance was to improve our lower bounds for $c_1(B(1, p), B_{n+k}(1))$. We are currently working on improving our upper bounds on $c_1(Z(1, p), Z_{n+k}(1))$.

Finally, we can now prove an energy-capacity inequality in the following situation.

Theorem 2.4. *Let (M, ω) be a symplectic manifold, and let $L \hookrightarrow M$ be a closed, embedded Lagrangian submanifold such that $\omega|_{\pi_2(M,L)} = 0$ and $\mu|_{\pi_2(M,L)} = 0$, μ the Maslov index. Let $U \subset M$ be a $2n$ dimensional symplectic submanifold of M with $U \cap L \neq \emptyset$. Denote by $E(L, U)$ the infimum of the Hofer norm over all Hamiltonian symplectomorphisms that displace L from U. Then $c_1(U, U \cap L) \leq E(L, U)$.*

3. Future collaboration with host institution

Our conjecture about the energy-capacity inequality for $c_1(M, L)$ remains open for cases not treated in Theorem 2.4 above, and, in addition, it is now reasonable to extend this conjecture to pairs (M, N), where N is coisotropic. We had initially hoped that for monotone Lagrangians with minimal Maslov number ≥ 2, when Lagrangian Floer homology is defined, we would be able to prove the energy-capacity inequality for $c_1(M, L)$ by analyzing the Lagrnangian boundary depth [2]. Unfortunately, we were not able to do so, and we currently believe that a new chain-level invariant is required. We have proposed one using chain-level PSS maps restricted to special cycles in the Morse chain complex. If it (or a variant) can be shown to be an invariant of the homotopy class rel endpoints of a path of Hamiltonian diffeomorphisms, it would have considerable independent interest as well. While this approach appears promising, much remains to be done before it is fully understood. Finally, the co-isotropic spectral invariants developed in [1] may also give us a tool to prove a co-isotropic energy-capacity inequality for certain pairs (M, N) as well.

4. Projected publications resulting from this grant

The extension of our earlier construction to coisotropic submanifolds will significantly strengthen an article that was in progress at the beginning of the visit. The theorem on the energy-capacity inequality above will form the core of a second article.

References

[1] Urs Frauenfelder and Peter Albers. Spectral invariants in Rabinowitz-Floer homology and global Hamiltonian perturbations. *Journal of Modern Dynamics*, 4(2):329–357, 2010.

[2] Michael Usher. Hofer's metrics and boundary depth. *Annales Scientifiques de l'École Normale Supérieure*, to appear.