CHERN-SIMONS GAUGE THEORY: 20 YEARS AFTER
SCIENTIFIC REPORT

Summary

This is the scientific report for the workshop *Chern-Simons gauge theory: 20 years after* held at the Max-Planck-Institute for Mathematics in Bonn from August 3 until August 7, 2009. The organizers were

- Benjamin Himpel, Universität Bonn, Germany,
- Hans U. Boden, McMaster University, Hamilton, ON, Canada,
- Jœrgen E. Andersen, Århus Universitet, Denmark,
- Atle Hahn, Universidade de Lisboa, Portugal.

The workshop brought together 72 researchers from Austria, Brazil, Canada, Denmark, France, Germany, Japan, Korea, Luxembourg, Netherlands, Poland, Portugal, Romania, Spain, Sweden, Switzerland, United Kingdom and the United States of America working in geometric topology, stochastical analysis and mathematical physics with a particular interest in Chern-Simons theory. Chern-Simons theory has developed in several different directions since its relation to knot theory had been discovered by Edward Witten 20 years ago. The workshop expanded the interaction and exchange of ideas between these communities of researchers. There were a total of five 60-minute talks and twenty-one 50-minute talks by internationally recognized experts, notably also by Witten himself. We left ample time for questions, problem sessions and informal discussion. An boat excursion on Wednesday afternoon and a social gathering on Thursday evening with some live music by workshop participants provided the change of scenery which is so often an additional boost for creativity. The workshop was the perfect opportunity for European researchers with an interest in Chern-Simons theory to connect with each other and internationally.

We are planning to publish workshop proceedings in 2011, which will not only document the workshop very well but is likely to have an impact on the research in Chern-Simons gauge theory. For more information on the workshop and, in particular, the upcoming proceedings see the workshop-website:

`http://www.hausdorff-center.uni-bonn.de/event/2009/gauge_theory/`

Scientific content

Let G be a semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. A connection A on the trivial G-bundle over a closed oriented 3–manifold M is a $\mathfrak{g}$-valued 1–form, i.e. $A \in \Omega^1(M;\mathfrak{g})$. Chern-Simons theory is a quantum field theory in three dimensions, whose action is proportional to the Chern-Simons invariant given in [7]

$$cs(A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

Witten introduced Chern-Simons theory to knot theory in 1989 [31], when he described for each integer level $k \in \mathbf{Z}$ an invariant of a link $L = (L_j)$ in a 3–manifold

M (and a list of finite-dimensional representations ρ_j of G associated to L_j) as the (non-rigorous) Feynman path integral

$$Z_k(M, L) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi k i\ cs(A)} \prod_j \mathrm{Tr}_{\rho_j}(\mathrm{hol}_A(L_j))dA,$$

where $\mathcal{A}$ is the space of G–connections, $\mathcal{G}$ is the group of gauge transformations. He interpreted these invariants using the axioms of topological quantum field theory (TQFT) as well as via an asymptotic expansion—the semiclassical approximation—by using the method of stationary phase. There have been several advances in understanding both approaches separately, notably the rigorous construction of the TQFT version by Reshetikhin and Turaev [27], the first computer calculations and the refinement of the semiclassical approximation by Freed and Gompf [9] and various recent work by Andersen, Hansen, Ueno and Takata [3, 5, 6, 4, 14, 15, 16, 17] in the realm of the asymptotic expansion conjecture for the Resehetikhin-Turaev invariants [26, Section 7.2]. However, there have been relatively few developments relating the TQFT version to the semiclassical approximation apart from the direct comparison for lens spaces for $G = SU(2)$ and some torus bundles over S^1 by Jeffrey [18], where she explicitly analyzed the asymptotic behaviour of the Reshetikhin-Turaev invariants and exhibited agreement in the leading term with the semiclassical approximation. One promising approach is to give a rigorous treatment of the Feynman path integral version using stochastical analysis via Fresnel integrals or Hida distributions, if possible, this would provide a strong link between the TQFT version and the semiclassical approximation. See [2, Section 10.5.5] by Albeverio, Høegh-Krohn, and Mazzucchi as well as [13] by Hahn for an overview.

In [31], Witten proposed a physical interpretation of the Jones polynomial using Chern-Simons theory, and in [19] Kashaev made the remarkable prediction that the volume of a hyperbolic knot K in S^3 is given by the limit of the colored Jones polynomial $J_N(K, q)$ of K. This is known as the volume conjecture, and the precise statement is that $\log |J_N(K, e^{2\pi i/N})|/N$ limits to $\frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K)$ as $N \to \infty$. (This conjecture can be extended to all knots by simply replacing hyperbolic volume with simplicial volume on the right hand side.)

Using $SL(2, \mathbf{C})$ Chern-Simons theory, Gukov conjectured in [11] that the asymptotic expansion of the colored Jones polynomial $J_N(K, q)$ as $N \to \infty$ and $q \to 1$ should be equal to the partition function of the $SL(2, \mathbf{C})$ Chern-Simons knot theory on $S^3 \setminus K$, and using this he derived a parameterized version of the volume conjecture which postulates a relationship between the family of limits of the colored Jones polynomial and the volume function Vol(K, u) on the character variety of the knot complement. The paper [12] provides further evidence for the generalized conjecture by studying some of the sub-leading terms in the asymptotic expansion of the colored Jones polynomial.

There is a strong connection between Witten's invariants and the so-called finite-type invariants. The first definition of a finite-type 3-manifold invariant was given by Ohtsuki [25], though by now many other useful and equivalent definitions have been given (see e.g. [10, 24]). The first nontrivial finite type 3–manifold invariant can be identified with the SU(2) Casson invariant [1, 30], and indeed all the terms in the stationary phase expansion of the Chern-Simons path integral give rise to finite type 3–manifold invariants. (See the papers [28, 23] for a mathematically rigorous definition of invariants arising from a stationary phase approximation, as well as proofs the invariants are of finite-type.)

A different direction of research was initiated by Taubes in 1990 [29]. He laid the groundwork for new topological invariants motivated by Chern-Simons theory by showing that the $SU(2)$ Casson invariant has a gauge theoretical interpretation as the Euler characteristic of $\mathcal{A}/\mathcal{G}$ in the spirit of the Poincaré-Hopf theorem, where he views the Chern-Simons invariant as a S^1-valued Morse function on $\mathcal{A}/\mathcal{G}$. Taubes realized that the Hessian of the Chern-Simons invariant and the odd signature operator coupled to the same path of $SU(2)$ connections have the same spectral flow. Floer extended this idea around the same time to instanton Floer homology [8], which has the $SU(2)$ Casson invariant as its Euler characteristic, by viewing the critical points of the Chern-Simons function as a $\mathbf{Z}/8$ graded Morse complex.

The Seiberg-Witten revolution in gauge theory shifted attention from instanton Floer homology to the more difficult monopole Floer homology, and progress in either of these versions of Floer homology has been vastly outpaced by the remarkable achievements in the closely related but much more accessible Heegaard-Floer homology introduced by Ozsváth and Szabó. Heegaard-Floer includes an entire package of homology theories which have become powerful tools in low-dimensional topology, and there has been some success in defining the corresponding pieces in instanton Floer and monopole Floer. Interest in these Floer homologies has been rekindled by the work [20, 21, 22] of Kronheimer and Mrowka; in [20] they give a thorough and rigorous treatment of monopole Floer theory, in [21] they define knot Floer homology using instantons, and in [22] they use sutured Floer homology to give a new and more direct proof of property P for knots, as well as the result, originally due to Ghiggini and Ni in the Heegaard-Floer setting, that shows that monopole Floer knot homology detects fibered knots.

In summary, Chern-Simons gauge theory is an active area of research, which has produced a lot of interesting questions and results, as well as initiated the development of entirely new fields.

Assessment and impact

The workshop had a relatively high number of “external” participants, i.e. people who participated without being registered. In fact, some of the talks were attended by up to 200 people whereas the number of registered participants was 72. Moreover, 32 of the registered participants did not ask for any financial support. This is not only a good sign for the relevance of (and the interest in) the workshop but it also suggests that the impact of the workshop was probably considerably higher than for an average workshop at a comparable financial level.

Among the “external” participants there was a relatively high number of PhD students and young PostDocs who should have benefitted from the chance of getting in contact with some of the leading figures in the field. Also the more senior participants should have appreciated the chance of establishing further contacts and of getting a better overview over the recent developments in the field (cf. the previous section of this report).

Finally, let us mention that the workshop proceedings will contain several strong papers with original research, for which an unusually high number of citations can be expected.

In summary we regard the workshop as very successful. Encouraged by this success Jørgen Andersen and Benjamin Himpel are planning to organize a workshop on a similar topic at Aarhus University next year, aimed at physicists and differential geometers/topologists with an special interest in gauge theory.

List of speakers and final program

Here is a list of the speakers and the titles of the corresponding talks[1]

(1) Dave Auckly: *Gauge-string duality and the structure of large rank Chern-Simons invariants*
(2) Dror Bar-Natan: *Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots*
(3) Chris Beasley: *Localization for Wilson Loops in Chern-Simons Theory*
(4) Dana Fine: *A geometric alternative to gauge fixing in Chern-Simons theory on* $S^1 \times \Sigma$
(5) Stavros Garoufalidis: *Chern-Simons theory and arithmetic*
(6) Enore Guadagnini: *Functional integration and abelian link invariants*
(7) Sergei Gukov: *Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group*
(8) Chris Herald: *An SU(3) Casson invariant for rational homology spheres*
(9) Kazuhiro Hikami: *WRT invariants and modular forms*
(10) Rinat Kashaev: *On rings associated with ideal triangulations of knot complements*
(11) Louis H. Kauffman: *Khovanov Homology and the Potts Model*
(12) Mikhail Khovanov: *Categorification of quantum groups*
(13) Paul Kirk: *Non-abelian representations, homology 3-spheres, and knot concordance*
(14) Albrecht Klemm: *Chern-Simons Theory and Topological String theory on non-compact Calabi-Yau manifolds*
(15) Christine Lescop: *On the cube of the equivariant linking pairing for closed 3-manifolds of rank one*
(16) Marcos Marino: *Chern-Simons theory, the* $1/N$ *expansion, and string theory*
(17) Gregor Masbaum: *Integral structures in TQFT and the mapping class group*
(18) Catherine Meusburger: *Getting physics from 3d gravity:* What does an observer in 3d gravity see?
(19) Aleksandar Mikovic: *Invariants of spin networks embedded in 3-manifolds*
(20) Hitoshi Murakami: *SL(2;C)-representations and asymptotic behaviors of the colored Jones polynomial of a knot*
(21) Robert Penner: *Fatgraphs and finite type invariants*
(22) Ambar Sengupta: *Functional Integrals in Low-Dimensional Gauge Theories*
(23) Albert Schwarz: *Generalizations of Chern-Simons theory*
(24) George Thompson: *Chern-Simons Theory On Seifert 3-Manifolds*
(25) Jonathan Weitsman: *Fermionization and Convergent Perturbation Expansions in Chern-Simons Gauge Theory*
(26) Edward Witten: *Branes and Quantization*

[1] the abstracts of the talks can be found on the webpage
`http://www.hausdorff-center.uni-bonn.de/event/2009/gauge_theory/abstracts/`

Chern-Simons Gauge Theory: 20 years after
3-7 August, 2009 in Bonn
Schedule of Events

Monday		Tuesday		Wednesday		Thursday		Friday	
10:00 - 10:05	**Opening**	09:00 - 10:00	**Khovanov**	09:00 - 09:50	**Lescop**	09:00 – 10:00	**Penner**	09:00 – 9:50	**Masbaum**
10:05 – 10:55	**Garoufalidis**	10:00 – 10:30	*Coffee*	09:50 – 10:40	**Guadagnini**	10:00 – 10:30	*Coffee*	09:50 – 10:40	**Hikami**
10:55 – 11:25	*Coffee*	10:30 – 11:20	**Mikovic**	10:40 – 11:10	*Coffee*	10:30 – 11:20	**Klemm**	10:40 – 11:10	*Coffee*
11:25 – 12:15	**Thompson**	11:20 – 12:10	**Herald**	11:10 – 12:00	**Weitsman**	11:20 – 12:10	**Auckly**	11:10 – 12:00	**Fine**
12:15 – 14:10	*Lunch*	12:10 – 14:10	*Lunch*	12:00 – 12:50	**Murakami**	12:10 – 14:10	*Lunch*	12:00 – 12:50	**Gukov**
14:10 – 15:00	**Kirk**	14:10 – 15:00	**Kauffman**	12:50 – 14:00	*Lunch*	14:10 – 15:00	**Beasley**		
15:00 – 16:00	**Marino**	15:00 – 16:00	**Kashaev**	14:00 – 19:00	*Excursion*	15:00 – 16:00	**Witten**		
16:00 – 16:30	*Coffee*	16:00 – 16:30	*Coffee*			16:00 – 16:30	*Coffee*		
16:30 – 17:20	**Meusburger**	16:30 – 17:20	**Sengupta**			16:30 – 17:20	**Bar-Natan**		
						17:20 – 18:10	**Schwarz**		
18:30	Casa del Gatto (Italian restaurant)	18:30	Mika Restaurant (Sushi Buffet)	19:00	Parkrestaurant Rheinaue (and „Bier Garten“)	18:30	Cassius Garten (Vegetarian self-service restaurant)		
						20:00	Social gathering at the Hausdorff Institute (HIM)* with live music, wine and pretzels		

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Bonn, May 31, 2010

Appendix: Full list of participants and their affiliations

(1) Sergio Albeverio, University of Bonn
(2) Joergen E. Andersen, University of Aarhus
(3) Dave Auckly, Kansas State University
(4) Dror Bar-Natan, University of Toronto
(5) Stefan Bauer, University of Bielefeld
(6) Chris Beasley, SUNY
(7) Stefan Behrens, University of Bonn
(8) Adara Blaga, West University of Timisoara
(9) Hans U. Boden, McMaster University
(10) Francesco Costantino, IRMA, University of Strasbourg
(11) Tien Cuong Dinh, Professor at Paris 6 University
(12) Jérome Dubois, Université Paris 7, Denis Diderot
(13) Magnus Engenhorst, Institut fr Mathematik, Universität Augsburg
(14) João Faria Martins, Centro de Matemática da Universidade do Porto
(15) Alexander Felshtyn, University of Szczecin and MPIM, Bonn
(16) Dana Fine, University of Massachusetts
(17) Stavros Garoufalidis, Georgia Institute of Technology
(18) Masha Gordina, University of Connecticut/University of Bielefeld
(19) Enore Guadagnini, University of Pisa
(20) Sergei Gukov, California Institute of Technology
(21) Atle Hahn, Universidade de Lisboa
(22) Luiz Hartmann, USP - Universidade de São Paulo
(23) Andriy Haydys, University of Bielefeld
(24) Chris Herald, University of Nevada
(25) Michael Heusener, Université Blaise Pascal Clermont II
(26) Kazuhiro Hikami, University of Tokyo
(27) Benjamin Himpel, University of Bonn
(28) Iulia Elena Hirica, University of Bucharest
(29) Fuji Hiroyuki, Nagoya University
(30) Lotte Hollands, University of Amsterdam
(31) Saeid Jafari, College of Vestsjaelland South (VUC)
(32) Franck Jedrzejewski, CEA (French Atomic Commission)
(33) Alexander Kahle, University of Göttingen
(34) Uwe Kaiser, Boise State University
(35) Rinat Kashaev, University of Geneva
(36) Louis H. Kauffman, Math Dept, University of Illinois at Chicago
(37) Gerald Kelnhofer, University of Wien
(38) Mikhail Khovanov, Columbia University
(39) Hoil Kim, Kyungpook National University
(40) Paul Kirk, Indiana University
(41) Albrecht Klemm, University of Bonn
(42) Valentin Krasontovitsch, University of Bonn (Math student)
(43) Markus Land, University of Bonn
(44) Jonatan Lenells, Cambridge University
(45) Christine Lescop, University of Grenoble I
(46) Poon Chuan Adrian Lim, University of Luxembourg
(47) Christoph Luedeling, University of Bonn

(48) Marcos Mariño, University of Geneva
(49) Cristina Martinez, Autonoma University of Barcelona
(50) Gregor Masbaum, Université Paris 7
(51) Gwenael Massuyeau, CNRS, University of Strasbourg
(52) Andrew McIntyre, Bennington College
(53) Catherine Meusburger, University of Hamburg
(54) Jouko Mickelsson, University of Helsinki and Royal Inst. of Technology
(55) Aleksandar Mikovic, Universidade Lusofona, Lisbon
(56) Hitoshi Murakami, Tokyo Institute of Technology
(57) Chiara Nappi, Institute for Advanced Study
(58) Robert Penner, University of California, Los Angeles
(59) Arturo Prat-Waldron, University of California, Berkeley and MPIM Bonn
(60) Justin Roberts, University of California, San Diego
(61) Nobuya Sato, Rikkyo University
(62) Albert Schwarz, UCDavis/MPIM
(63) Ambar Sengupta, Louisiana State University
(64) Jan Swoboda, MPIM, Bonn
(65) George Thompson, International Centre for Theoretical Physics ICTP
(66) Giorgio Trentinaglia, Georg-August-Universität Göttingen
(67) Alessandro Valentino, University of Göttingen
(68) Roland van der Veen, University of Amsterdam
(69) Jonathan Weitsman, Northeastern University
(70) Edward Witten, Institute for Advanced Study
(71) José Miguel Zapata Rolón, University of Bonn
(72) Lucy Liuxuan Zhang, University of Toronto