Final Report of the short visit to CIRGET (UQAM, Montreal)

Complex deformations, stable bundles and the Strominger System

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1 Purpose of the visit

During the visit of 15 days (31st May - 14th June, 2013) to CIRGET (UQAM, Montreal) to collaborate with Carl Tipler, we addressed two different problems. As a common ground, both problems involve deformations of complex manifolds endowed with stable bundles, and partial differential equations motivated by String Theory and Mathematical Physics.

A) Applications of the deformation Theory for the CKYM equations: In recent work with Carl Tipler [4], we have defined a deformation theory for pairs (X, E), given by a Kählerian complex manifold X and a holomorphic vector bundle E over it. For this, we use gauge theoretical equations introduced during my Phd Thesis [3], known as the Coupled Kähler–Yang–Mills equations [1]. As a continuation of this research, during the visit we studied applications of this theory to stable bundles over abelian varieties and Calabi-Yau 3-folds.

B) Geometry of heterotic string compactifications: The Strominger System (ST for short) is a coupled system of partial differential equations that arises naturally in compactifications of the heterotic string theory [7]. It provides a generalization of the Kähler Ricci-flat equation for (conformally balanced) hermitian metrics on Calabi-Yau manifolds (complex manifolds with trivial canonical bundle). Its study has been proposed by S.-T. Yau to understand moduli spaces of Calabi-Yau manifolds in complex dimension 3. During the visit to CIRGET, jointly with Carl Tipler we addressed the ST problem (existence and uniqueness) and also the problem of constructing a local parameter space for the moduli of solutions of ST using Elliptic Operator Theory.

2 Description of the work carried out during the visit

A) During the visit we have studied the deformation complex studied in [4] in the case of holomorphic vector bundles over polarized complex manifolds that do not admit any non-vanishing Hamiltonian holomorphic vector field (e.g. Calabi-Yau manifolds). When the bundle admits a Yang-Mills connection compatible with the holomorphic structure, the deformation complex admits a particularly explicit description, that lead us to an effective criterion to decide whether a stable bundle over a Calabi-Yau manifold admits stable deformations.

B) Concerning this problem, during the visit we have been mainly focused on the study of the anomaly equation

$$dd^c\omega = \operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F$$

appearing in the Strominger System. This equation has been little studied in the literature from an analytical point of view. For this, we have relied on the geometric interpretation of this equation due to D. Freed [2], as the existence of a flat connection on a determinant line bundle over a (suitable) moduli space of smooth surfaces on the Calabi-Yau. We expect that further studies of this gauge-theoretical interpretation of the anomaly equation may have future applications in the existence problem for ST.

3 Description of the main results obtained

The main result obtained concerns problem A), and is (roughly) the following.

Theorem. Let X be a Kähler manifold and E holomorphic vector bundle over it, with fixed hermitian metric h and Chern connection A. Given a small deformation γ of X compatible with the Kähler form (see [4]) we have

1. if the Atiyah class $0 \neq [F_A^\gamma] \in H^2(\operatorname{ad} E)$ then the fibre $p^{-1}(\gamma)$ over γ of the space of joint deformations $H^1(X, L_\omega^*)$ of (X, E) compatible with the Kähler form is empty.

2. if $[F_A^\gamma] = 0$ and (X, ω) admits no non-zero holomorphic (complex) Hamiltonian vector fields, then there exists an injection

$$\{\beta \in \Omega^{0,1}(\operatorname{ad} E) : \Delta_A \beta = 0\} \to p^{-1}(\gamma) \subset H^1(X, L_\omega^*)$$

3. Furthermore, if A is Yang-Mills then $[F_A^\gamma] = 0$ implies $F_A^\gamma = 0$ and in this case, the previous map is given by

$$\beta \mapsto (\gamma, \beta)$$

4. Furthermore, if A is Hermitian-Yang-Mills, the previous map induces an equivariant bijection

$$H^1(X, \operatorname{ad} E) \to p^{-1}(\gamma) \subset H^1(X, L_\omega^*).$$

An element (γ, β) in the image is unobstructed if and only if

$$[\beta, \beta] = 0.$$

The importance of this result is that it provides a completely explicit description of the deformation space for pairs (X, E) in terms of deformations of the base and deformations of the bundle E (with X fixed). This has lead us to recover some results of D. Huybrechts [5] for the deformation of $\mathcal{O}_X \oplus TX$ on a Calabi-Yau 3-fold. We expect this result to have original interesting applications.

4 Projected publications/articles resulting or to result from your grant

As a result of the visit, we plan to improve the preprint [4] with the new result and some applications derived from it.

References

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