Scientific Report

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ITGP Project "Quantisation of dissolving vortices"
Short Visit to Prof. José Mourão, IST, Lisbon
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1 Purpose of the visit

This short visit was aimed at familiarising the **host**, Prof. José Mourão, with recent results about the vortex equations (on line bundles over Riemann surfaces) and the associated Kähler metrics on their moduli spaces, and conversely the **visitor**, Dr Nuno Romão, with current research on geometric quantisation and Kähler geometry in the host's group at IST, with a view towards identifying topics of collaboration.

2 Description of the work carried out during the visit

- The main activity of this visit was a series of discussions focused on the topics mentioned above, in which Dr Nuno Romão (the visitor), Prof. José Mourão (the host) and Dr João Pimentel Nunes (a collaborator of the host at IST) took part. Some results of these discussions will be summarised in Section 3.
- The visitor gave a seminar entitled "Vortices and Jacobian varieties" on the 3rd of May 2012, in the series *Geometria em Lisboa* hosted by the Department of Mathematics of IST. In this seminar, aspects of the complex geometry of the L^2-metrics on vortex moduli spaces were presented to a broader audience.
- The visitor also took the opportunity to discuss topics related to the vortex equations with Dr João Baptista, another specialist on vortices who is starting a post-doc position at the Department of Mathematics of IST.

3 Description of the main results obtained

3.0 The main subject of our discussions was the possibility of applying the complex geometry of the Abel–Jacobi map

$$\mathrm{AJ} : \Sigma^{(d)} \to \mathrm{Jac}(\Sigma)$$

to the quantisation of the moduli space of d vortices in a line bundle over a closed Riemann surface Σ of genus $g > 0$. Specifically, this geometry provides an approximation to the L^2-geometry of the moduli space of vortices $\mathcal{M}_d \cong \Sigma^{(d)}$ on Σ of genus $g > 1$ in a 'dissolving limit' studied in recent joint work of the visitor with Manton [6]: the Kähler structure

ω_{L^2} on the moduli space then converges to the pull-back $\mathrm{AJ}^*\Omega_j$ of the polarisation of the Jacobian variety of the surface associated to the complex structure j on Σ. In reference [6], the focus was on the most interesting case $d < g$, for which AJ is not surjective. Then one is interested in the (curved) geometry on the subvariety $W_d = \mathrm{AJ}(\Sigma^{(d)})$ which is induced by the flat metric associated to the polarisation and can be regarded as a generalisation of the canonical Bergman metric on Σ.

From the point of view of quantisation, the present context fits well into the problem of quantising singular phase spaces, since W_d will typically have singularities if g and d are sufficiently close. Then one should be able to equip W_d with the structure of a stratified Kähler space, on which one can discuss quantisation at the level of the algebras of functions. Given such a structure, the occurrence of singularities in the phase space is not an unsurmountable obstacle, since one can reformulate certain linearised structures needed in geometric quantisation (e.g. tangent spaces) in terms of the Lie–Rinehart algebras defined from the underlying Poisson structures and their associated modules [4].

3.1 One observation that emerged from our discussions is that the problem of quantisation of dissolved vortices can naturally be given a second-quantisation perspective in the context of the geometric quantisation of (Σ, ω_Σ) (a closed Riemann surface with a fixed area form). In fact, the Hermitian bundle $L \to \Sigma$ which is fixed to write down the vortex equations, together with the connection $\mathrm{d}_A = \partial_A + \bar\partial_A$ associated to a dissolved vortex, can be thought of as prequantisation data for the symplectic structure ω_Σ. The definition of a dissolved vortex requires the holomorphic structure $\bar\partial_A$ on the line bundle associated to this connection to have nontrivial kernel, and this kernel corresponds to a nontrivial space of (polarised) wavefunctions ϕ, i.e. quantum Hilbert spaces associated to (Σ, ω_Σ). So the moduli space of dissolved vortices can be described as a bundle of spaces of rays (quantum states) $\mathbb{P}(\ker \bar\partial_A)$ over the space of possible prequantisation data (parametrised by $W_d \subset \mathrm{Jac}(\Sigma)$) compatible with a fixed Hermitian metric on L. The problem of quantising dissolved vortices can be then thought of as a second quantisation of the wavefunctions arising from this constrained geometric quantisation of ω_Σ: one treats the space of (rays of) wavefunctions as a phase space that feeds again into the geometric quantisation machinery.

3.2 The geometry of dissolving vortices provides an approximation to the L^2-geometry where the Kähler structure can be described concretely and exhibits the dependence on the complex structure of the curve very explicitly. In the context of geometric quantisation of Kähler polarisations, this leads to interesting directions to probe several questions that for the moment are inaccessible for the general L^2-metrics on $\Sigma^{(\mathrm{d})}$.

One interesting problem is the calculation of the volumes of the vortex moduli spaces and compare them with dimensions of spaces of holomorphic sections (i.e. quantum Hilbert spaces in Kähler quantisation). A straightforward calculation yields that for a surface of genus g, the moduli space W_d of d dissolved vortices has symplectic volume $(2\pi)^d \frac{g!}{(d-g)!d!}$ for $1 \le d \le g$, irrespective of the complex structure. By a well-known physical argument, one then expects semiclassically that the quantum Hilbert spaces should have dimension $\binom{g}{d}$. A first (and maybe naive) approach to proceed with the quantisation is to take the Θbundle on the Jacobian, pull it back to W_d and define the Hilbert space to be the space of regular sections of this bundle. In this setting, it is trivial to check that this "semiclassical prediction" actually holds for $d = g$, for which $W_g = \mathrm{Jac}(\Sigma)$ and the Hilbert space consists of multiples of the Riemann Θ-function, and also in the case of an elliptic curve $g = 1$. A natural way of understanding the semiclassical regime is to devise a limiting process where one rescales the symplectic form by an integer k and then renormalise the volume by dividing by k^d. A standard application [11] of the Grothendieck–Riemann–Roch theorem then gives that the semiclassical prediction holds at least when $k \to \infty$. If the semiclassical prediction does not hold in general, an interesting question to ask is how the ratios of the dimensions of the Hilbert space to the normalised volumes (as k increases) vary with the complex structure, and if so to try to understand how the ratios behave in the moduli space of curves — for example, one might be able to identify complex structures on a compact surface that behave "classically" in this respect, make precise the notion that some complex structures behave more classically than others, and how the existence of singularities in the phase spaces affects this behaviour. We presently need some extra input from algebraic geometry to be able to answer these questions.

For general $d < g$, one knows that W_d may have singularities (in particular, it always does if $d > g/2$), and that the structure of the singular locus depends on the complex structure. Strictly speaking, geometric quantisation is not well defined in the singular case, but one may still define a refinement of it following the scheme described in [4]. One basic aim of our project consists of understanding how the presence of singularities introduces extra structure on top of the naive version of Kähler quantisation described above. Following Huebschmann, one starts with a (degenerate) Poisson structure on the algebra of functions on the phase space – in our context, one would take the canonical symplectic Poisson structure defined on generators of the algebra of local functions on the Jacobian (local coordinates), and then subject them to the relations imposed by the local description of the subvariety W_d; this produces a so-called Lie–Rinehart algebra [4]. Then the process of prequantisation is described in terms of a "prequantum module [4] for this algebra. The structure of stratified Kähler space on W_d then yields a co-stratified Hilbert space of polarised sections of the prequantum module; the co-stratification can be regarded as extra information that reflects the stratification of the singular phase space of the classical theory. In the context of dissolved vortices, we would like to understand this co-stratification in algebraic-geometric terms. The simplest nontrivial example of the singular situation corresponds to taking $d = 2$, $g = 3$ and Σ hyperelliptic. The image W_2 is then just (a translation of) the Θ-divisor. The structure of this particular Θ-divisor is known quite explicitly; it is a surface with a double point — part of this is very classical, but further progress has been made more recently fostered by approaches to the moduli space of curves motivated by the study of integrable hierarchies [3, 8].

3.3 Another topic of focus of our discussions was the possibility of exploring the technique of Gromov–Hausdorff collapse of Kähler structures, in which the group of Prof. Mourão has gained considerable expertise, to obtain simplified descriptions of moduli spaces of vortices and their metrics. One case of collapse that should be amenable to study in the context of the dissolving limit is the moduli space of $(g-1)$-vortices on a surface of genus g. Then the moduli of vortices is approximated by (a translate of) the Θ-divisor of the Riemann surface. The degeneration of Abelian varieties described in [1, 7, 2] as the complex structure approaches the boundary of the Siegel half-space produces a Gromov–Hausdorff collapse of the Jacobian of the surface to a real torus of dimension g, and with it a degeneration of the Θ-divisor, which is to some extent captured in terms of tropical geometry. It would be of interest to understand this process in terms of the degeneration of the Riemann surface itself, and how features of the quantisation, such as the relation between symplectic volumes and dimensions of Hilbert spaces and co-stratification described above, behave under it. Another promising setup to study Gromov–Hausdorff collapse of vortex metrics arises for vortices on the Euclidean plane, for which Manton and Speight [5] have obtained explicit results on the L^2-metrics in the asymptotic regime of large separation. One interesting starting point would be to parametrise a polynomial family of complex structures on $\mathbb{C}$ and understand how the calculations of Manton and Speight deform under this family; it is expected that the second-order ODEs from which the asymptotic metrics are deduced will regulate the deformations of the metric, and maybe their behaviour under Gromov–Hausdorff collapse. Another interesting problem consists of exploring the natural Ginzburg–Landau potential on the moduli space, whose asymptotics for well-separated vortices are also known [9], as a natural complexifier for the complex-time Hamiltonian flow discussed in [10] to produce a natural class of adapted complex structures for the asymptotic Kähler metric.

4 Future collaboration with the host institution

The intervenients in this research visit have agreed to maintain contact on developments of the discussions detailed above. Dr João Pimentel Nunes will be visiting Bonn in July 2012, which will be the next opportunity to continue our discussions. It will be decided in the near future whether a second visit of Dr Nuno Romão to Lisbon will be profitable to develop joint work on the ideas described in Section 3.

5 Projected publications

This report concerns exploratory scientific research at a very early stage, and there is no definite plan as yet regarding publication.

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