HIGGS BUNDLES, SURFACE GROUP REPRESENTATIONS AND HERMITIAN SYMMETRIC SPACES

ROBERTO RUBIO

1. Purpose of the visit

The purpose of the visit was the study of the moduli space of polystable G-Higgs bundles with maximal Toledo invariant when G is the non-compact real group of isometries of a Hermitian symmetric space of non-tube type. In this case, a rigidity phenomenon has been observed in the classical cases ([2]). All objects are strictly polystable and the moduli space can be expressed as a fibration. However, there is no general proof of this fact as far as we know. This proof would not rely on classification theory and would point out the key feature of these groups which make the rigidity and the fibration work.

The concept of Higgs bundle over a Riemann surface X was originally introduced by Nigel Hitchin in 1987 ([3]). Defined into different contexts, the moduli of Higgs bundles tie together the moduli space of representations of the fundamental group $\pi_1(X)$ with the moduli space of solutions to the Hitchin equations.

We focus on the case of a real semisimple non-compact Lie group G (with maximal compact subgroup H). In this case, a G-Higgs bundle consists of a principal $H^{\mathbb{C}}$-bundle E together with a holomorphic section $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ (where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the Cartan decomposition and K is the canonical bundle).

The assumption of G/H being an irreducible Hermitian symmetric space gives some valuable pieces of extra information, such as the decomposition of the Higgs field into two parts $\varphi = (\beta, \gamma)$ (given by the decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$). In this general setting, we define a Toledo character from the isotropy representation and the dual Coxeter number N of the group. First we define it in the Lie algebra as a sum of positive non-compact roots (Δ_Q^+):

$$d\chi_T = \frac{2}{N} \sum_{\alpha \in \Delta_Q^+} \alpha$$

and then show that lifts to a character of the group. The Toledo invariant d is defined as the degree of the bundle associated to this Toledo character,

$$d = \deg(E(\chi_T)).$$

We obtain a Milnor-Wood type inequality finer than the usual one, for a properly defined rank:

$$-\operatorname{rk}(\beta)(g-1) \leqslant d \leqslant \operatorname{rk}(\gamma)(g-1).$$

The moduli space can be split according to the Toledo invariant, and the Milnor-Wood type inequality tells us that there are only a finite number of these pieces, as the ranks of β and γ are bounded by $\operatorname{rk}(G/H)$. We focus on the maximal case, $d = \pm \operatorname{rk}(G/H)(g-1)$.

2. Description of the work carried out

During the four weeks of collaboration with Professor Olivier Biquard we have identified the ingredients needed to state the phenomenon of rigidity in a classification-independent way. We have thoroughly studied the paradigmatic example of $\mathrm{SU}(p,q)$.

Let G be a non-compact real Lie group of Hermitian non-tube type. Every polystable G-Higgs bundle is strictly polystable and its structure group reduces to the normalizer of the subtube group $N = N_{H^{\mathbb{C}}}(H_T^{\mathbb{C}})_0$.

We consider the following groups:

$$C := C_{H^{\mathbb{C}}}(H_T^{\mathbb{C}}) \qquad C_G := C_{G^{\mathbb{C}}}(G_T^{\mathbb{C}}) \quad C_G^{ss} = [C_G, C_G]$$

$$Z := Z(H^{\mathbb{C}}) \qquad \Gamma := Z(G_T^{\mathbb{C}}) \quad \Gamma' = C_G^{ss} \cap Z(C_G)$$

We know that $N = CH_T^{\mathbb{C}}$ and $\frac{N}{H_T^{\mathbb{C}}} \cong \frac{C}{Z}$.

The group N injects into

$$\frac{N}{H_T^{\mathbb{C}}} \times \frac{N}{C_G^{ss}}$$

and we denote by Q its quotient.

The following diagram is proved to commute:

$$\begin{array}{ccc} H^1(X,N) & \xrightarrow{\pi} & H^1(X, N/H_T^{\mathbb{C}}) \\ \downarrow & & \downarrow{\scriptstyle c_1} \\ H^1(X, N/C_G^{ss}) & \xrightarrow{c_2} & H^1(X,Q) \end{array}$$

3. Description of the main results obtained

Up to checking some technical details, the main result is established as follows:

Theorem. *The moduli space $\mathcal{M}_{max}(G)$ is a fibration with base the moduli space of holomorphic $N/H_T^{\mathbb{C}}$-bundles $M(N/H_T^{\mathbb{C}})$ and fibre isomorphic to the moduli space of G_T-Higgs bundles, $\mathcal{M}(G_T)$.*

As it is seen from the diagram above, the N-bundles project to $N/H_T^{\mathbb{C}}$-bundles. The fiber over $F_0 \in H^1(X, N/H_T^{\mathbb{C}})$ will consist of

$$\{E \in H^1(X, N/C_G^{ss}) \,|\, c_2(E) = c_1(F_0)\},$$

which together with the Higgs field, which lives in its isotropy representation, has to be shown isomorphic to $\mathcal{M}(G_T)$.

One of the main issues that remains is that of stability. But we feel very confident about it, as it has been studied with some detail in proving the Cayley correspondence for the tube-type case.

4. Future collaboration with host institution

I started to collaborate with Professor Olivier Biquard during my research stay in the IMJ from 15 September to 16 December of 2009. This grant has given us the opportunity to continue this work. The research conducted almost finishes a first part of our scientific project, but it remains to use the results obtained to deduce some properties about the topology of the moduli space, namely, the number of connected components. On the other hand, the techniques developed may be applied to non-maximal cases to obtain some further information. Therefore, the collaboration will continue in the future.

5. Projected publications

The research conducted will be part of the future research paper about Higgs bundles and Hermitian symmetric spaces. It will be authored by Olivier Biquard, Oscar Garcia-Prada and Roberto Rubio. The grant of the ESF will be conveniently acknowledged in the introduction.

References

[1] O. Biquard, O. Garcia-Prada and R. Rubio, Higgs bundles and Hermitian symmetric spaces, *in preparation.*

[2] S. Bradlow, O. Garcia-Prada and P.B. Gothen, Maximal surface group representations in isometry groups of classical hermitian symmetric spaces, *Geometria Dedicata* (1) **102** (2006).

[3] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55** (1987), 59–126.