Scientific Report
Young European Probabilist 2012
Two-dimensional statistical mechanics

1 Summary

This workshop is the ninth in a successful series of Young European Probabilists (YEP) meetings taking place at Eurandom in the past years. Traditionally, these meetings bring together junior probabilists from all over Europe working in a particular area of research, allowing them to present and discuss their recent work. This year, the main focus of the workshop was on "Two dimensional statistical mechanics". Since the introduction of SLE processes by Oded Schramm and the work of Stanislav Smirnov on conformal invariance of percolation, the area of two dimensional statistical mechanics has become one of the most exciting and active subfields of modern probability. Substantial progress has been made since then, especially in understanding two-dimensional phase transitions, and answers to many fundamental and long standing questions have been found. Apart from its direct relevance to physics, this new and exciting area of mathematics is underlied by deep connections between probability, combinatorics, the theory of conformal maps, and many other parts of mathematics, and continues to be a great source of mathematical discoveries.

The workshop focused on recent progress in the field, featuring three 6 hrs mini-courses by Nicolas Curien, Dmitry Chelkak and Gabor Pete, as well as a number of talks by young European researchers working on different models of two-dimensional statistical mechanics. Important open problems were extensively discussed during special open problem sessions at the workshop. Relatively long lunch breaks facilitated scientific discussions and collaborations between the participants.

2 Scientific content and discussions

The workshop started by a introductory talk by Ariel Yadin. This talk, entitled *A Brief Introduction to SLE*, was aimed to provide a brief introduction to Schramm-Loewner Evolution, SLE for short. This object is central in planar statistical physics and appeared in several other talks during the week. While it is now a standard object, it was still necessary to remind its definition and some of its specific features. jumping forward 70 years to the domain Markov property and conformal invariance. We will view these notions through the first process considered by Oded Schramm when he introduced SLE: loop-erased random walk.

The rest of the week was organized as follows. Mornings were dedicated to mini-courses, each session being 90 minutes long (each speaker had 3 sessions). In the afternoon, 60 minutes talks alternated with short-talk session (15 minutes each) and open problem sessions. Let us now describe the content of the talks.

2.1 Mini-Courses

The first mini-course, entitled *Discrete complex analysis on the microscopic level: conformal invariants without conformal invariance*, was given by D. Chelkak (PDMI RAS & Chebyshev Lab). The speaker developed a theory of discrete holomorphicity and discrete harmonic measures. The main object of this theory is the study of the critical Ising model. This classical model of statistical physics, when considered on the square lattice, is conformally invariant in the scaling limit. This result, due to D. Chelkak and Fields Medallist S. Smirnov, enables to study the phase transition of the Ising model (in particular, the computation of the critical exponents). In order to prove conformal invariance of the model, one needs to introduce pre-holomorphic observables of the discrete model. It becomes fundamental to develop a discretization of the theory of conformal maps. This mini-course can therefore be understood as a toolbox, which will be useful for researchers trying to prove conformal invariance of lattice models.

The second mini-course, entitled *What is a random planar geometry*, was given by N. Curien (ENS Ulm). This course dealt with another major branch of today's planar statistical physics, namely the theory of random planar maps. This theory has been developed over the last years in part motivated by the theory of two-dimensional quantum gravity. In particular, very recently, Le Gall and Miermont independently showed that a large class of random planar maps admits a continuum limit, a compact random surface called the Brownian map. In this minicourse, the speaker discussed uniform infinite planar maps which appear as local limits of uniform planar

maps as their sizes go to infinity, and can be seen as a random infinite planar graph.

The last mini-course, entitled *Dynamical and near-critical percolation: many questions and many answers*, was given by G. Pete (University of Budapest). It mostly discussed dynamical percolation. Static percolation is a model of random graph, for which a site of a lattice is either open or closed with probability p or $1-p$. On the triangular lattice and for $p = 1/2$, the percolation model is critical. Dynamical percolation is the natural time evolution with critical percolation as stationary measure: every site of the lattice is switching between open and closed according to an independent exponential clock. While at typical time, there is no infinite cluster of neighboring open sites, there can a priori be exceptional times for which an infinite cluster appears. Motivated by these questions, Schramm and Steif proved the existence of exceptional times for the triangular lattice. Garban, Pete and Schramm developed a much deeper theory of the model, based on the conformal invariance of critical percolation proved by Smirnov (this proof is related to the first course given by Chelkak). In particular, they studied in details the set of exceptional times. The speaker described several aspects of this theory. He also discussed more recent works dealing with the near-critical percolation and Ising-related dynamics. In this later case, he harnessed results closely related to Chelkak-Smirnov theorem of conformal invariance for the critical Ising model mentioned above.

2.2 Talks

Conformal invariance and SLE The talk of K. Izyurov, entitled *Conformal invariance of spin correlations in the Ising model*, and the talk of K. Kytöla, entitled *Interface in critical 2D Ising model with plus-minus-free boundary conditions*, were both dealing with the conformal invariance of the critical Ising model.

The first one described the recent proof of conformal invariance of spin correlations. The main tools are discrete holomorphic spinor observables and results on convergence of solutions of corresponding discrete boundary value problems to natural continuum counterparts. This talk illustrates the relevance of a theory of discrete holomorphic functions, and is therefore very much related to Chelkak's minicourse.

The second talk dealt with another aspect of conformal invariance of the critical Ising model. While Izyurov's result discussed spin correlations, Kytöla's result dealt with the geometry of interfaces in the model. More precisely, it showed that an interface in the low temperature expansion with plus-minus-free boundary conditions converges in a limit to a variant of SLE(3) called the dipolar SLE(3).

In his talk, entitled *Annulus crossing and regularity of random curves*, A. Kemppainen presented an important precompactness result for interfaces in lattice models. This result was for instance used in Kytöla's talk to prove convergence to dipolar SLE(3). Roughly speaking, if a desired a priori estimate holds for a family of probability measures on the space of curves, then this family is supported on curves which are Hölder regular as curves and as Loewner chains. This framework can be used in order to prove the existence of subsequential limits for random curves. It is a fundamental part of the proof that some given sequence of random curves converges to a Schramm-Loewner evolution (SLE).

D. Beliaev's talk (*Two-point Schramm's formula and $SLE_{8/3}$ bubbles*) dealt with a continuous question on SLE processes. He proved a formula for the probability that the chordal $SLE_{8/3}$ path passes to the left of two points in the upper half-plane (this formula was predicted by Simmons and Cardy).

Another talk was very much related to critical lattice models. Ioan Manolescu discussed *Bond Percolation on Isoradial Graphs*. He explained that the star-triangle transformation can be used in order to prove universality of alternating arm-exponents (assuming they exist) for a wide class of bond percolation models, under some mild conditions. This result will be very useful once bond percolation will be proved to be conformally invariant.

Fluctuations of interfaces in off-critical models Two talks were dealing with models away from the critical point. L. Coquille, in her talk *Gibbs measures of the 2d Ising and Potts models*, presented a new proof of the celebrated Aizenman and Higuchi result on Gibbs measures for the low-temperature planar Ising model. The main ingredient is a study of fluctuations of interfaces in the low-temperature regime via the Ornstein-Zernike theory. This theory is not specific to the two-dimensional case, but its application to interfaces is.

Alan Hammond's talk on *Boundary roughness in supercritical FK percolation models* also harnessed a Ornstein-Zernike theory to describe a subcritical percolation cluster conditioned to be large. The cluster obtained has a macroscopic profile approximating the Wulff shape, and its fluctuations can be studied.

Dimers Another part of planar statistical physics is the study of dimers models. This important domain was represented by two talks. Anthony Metcalfe described *Universality problems relating to lozenge tilings of a hexagon*. Zhongyang Li generalized in her talk *1-2 Model, Dimers and Clusters* the Fisher transformation to a class of models called 1-2 models.

Miscellaneous François Simenhaus described a recent striking result on the Ising Glauber dynamics at zero temperature. Vincent Tassion discussed *The critical value function in the divide and color model.* The divide and color model is a simple and natural stochastic model for dependent colorings of the vertex set of an infinite graph. The speaker extended the RSW and coarse-graining theories available for percolation to this model in order to obtain fundamental results on the phase transition.

2.3 Short talks

Several short talks were presented. The connection to the main subject of the conference was not mandatory. We mention briefly the subject of these talks. S. Aumann exposed a result on the near critical percolation extending Nolin and Werner's result on mutual singularity of two different nearcritical percolation scaling limits. I. Kortchemski discussed models of random non-crossing configurations, proving a universality result on their scaling limits. M. Lis discussed a result very much related to the theme of the conference. He presented how to compute the critical temperature of the 2D Ising model via the combinatorial method. This method can be compared to the Kasteleyn approach to dimers, and invoke the Kac-Ward matrix. T. Muller described a continuum bootstrap percolation model. He extended the result of Aizenman and Lebowitz to this model. F. Nardi described metastability for Kawasaki dynamics at low temperature with two types of particles. R. Tanaka discussed some properties of random walks on Cayley graphs of groups of polynomial growth. Finally, C. Temmel presented results on stochastic domination of general Bernoulli product fields.

2.4 Open problem sessions

In addition to the talks, two open problem sessions were organized. These sessions were a great success (originally, only one was scheduled, but the number of open questions pushed us to organize a second session). This success bears witness to the dynamism of the field. Young mathematicians already had the ability to present interesting and difficult open problems, some of which will hopefully be solved by members of the audience in the near future.

2.5 Discussions

Let us mention that a long lunch break enabled participants to discuss together.

3 Assessment of the results and impact of the event on the future directions of the field

The meeting brought together young researchers from all over Europe working in various areas of two dimensional statistical mechanics. It was especially intended as a sourse of education and encouragement for researchers at the earliest stage of their career. These two important goals have been successfully achieved during the meeting. The minicourses, delivered by excellent speakers and prominent researchers, provided a detailed introduction into the most recent advances in the field and facilitated lively discussions among the participants. Wide range of topics of invited lectures reflected the richness and the mathematical beauty of the field and stimulated exchange of ideas between researchers working in different subfields of two dimensional statistical mechanics. Most importantly, the workshop created an inspiring atmosphere for youngest participants to discuss their research and first results with a bit more senior colleagues. This is certainly a valuable experience for those who is going to contribute to the field in the near future.