The period spent at UCD has brought the research to the following step forward in the investigated problem. We have identified a method for non independent samples that is able to apply for the type of data that we are given. Here is a short report of the method.

Let consider the Hamiltonian for R samples each long L

$$H_N(m_1,\dots,m_R) = -N\Big(\frac{J}{2R^2}\sum_{l,s=1}^{R} m_l m_s + \frac{h}{R}\sum_{l=1}^{R} m_l\Big) \tag{1}$$

The joint probability is:

$$P(m_1,\dots,m_R) = \sum_{\hat{\sigma}} \frac{e^{-H_N(m_1,\dots,m_R)}}{Z} = \frac{1}{Z}\prod_{l=1}^{R} A_{m_l} e^{-H_N(m_1,\dots,m_R)} \tag{2}$$

Our aim is to identify J e h to maximise the joint probability. We have:

$$\ln P(m_1,\dots,m_R) = \sum_{l=1}^{R} \ln(A_{m_l}) - H_N(m_1,\dots,m_R) - \ln Z \tag{3}$$

and the partial derivatives are

$$\begin{aligned} \frac{\partial \ln P(m_1,\dots,m_R)}{\partial h} &= \frac{N}{R}\Big(\sum_{l=1}^{R} m_l - \frac{1}{Z}\sum_{m_1,\dots,m_R}\prod_{l=1}^{R} A_{m_l} e^{-H_N(m_1,\dots,m_R)} \sum_{l=1}^{R} m_l\Big) \\ \frac{\partial \ln P(m_1,\dots,m_R)}{\partial J} &= \frac{N}{2R^2}\Big(\sum_{l,s} m_l m_s - \frac{1}{Z}\sum_{m_1,\dots,m_R}\prod_{l=1}^{R} A_{m_l} e^{-H_N(m_1,\dots,m_R)} \sum_{l,s=1}^{R} m_l m_s\Big) \end{aligned} \tag{4}$$

Remembering that

$$\sum_{m_1,\dots,m_R}\prod_{l=1}^{R} A_{m_l} e^{-H_N(m_1,\dots,m_R)} = \sum_{\sigma} e^{-H_N(\sigma)} \tag{5}$$

we have the maximum condition met when:

$$\begin{cases} \sum_{l=1}^{R} m_l = \langle \sum_{l=1}^{R} m_l \rangle_{BG} = R\langle m_l\rangle_{BG} \\ \sum_{l,s} m_l m_s = \langle \sum_{l,s} m_l m_s \rangle_{BG} = R\langle m_l^2\rangle_{BG} + R(R-1)\langle m_l m_s\rangle_{BG} \end{cases} \tag{6}$$

Then

$$\begin{cases} \langle m_l\rangle_{BG} = \frac{1}{R}\sum_{l=1}^{R} m_l \\ \langle m_l^2\rangle_{BG} + (R-1)\langle m_l m_s\rangle_{BG} = \frac{1}{R}\sum_{l,s} m_l m_s \end{cases} \tag{7}$$

The way the empirical data are related to h and J is, given that $\langle m_1\rangle_{BG} = \langle m_2\rangle_{BG} = \cdots = \langle m_R\rangle_{BG}$,

$$\lim_{N\to\infty} \langle m_1\rangle_{BG} = \mu \tag{8}$$

where $\mu = \tanh(J\mu + h)$ By consequence:

$$\frac{\partial}{\partial h}\langle m_1\rangle_{BG} = \frac{\partial}{\partial h}\langle m_2\rangle_{BG} = \cdots = \frac{\partial}{\partial h}\langle m_R\rangle_{BG} \tag{9}$$

and

$$\lim_{N\to\infty} \frac{\partial}{\partial h}\langle m_1\rangle_{BG} = \chi = \frac{1-\mu^2}{1-J(1-\mu^2)} \tag{10}$$

where:

$$\frac{\partial}{\partial h}\langle m_1\rangle_{BG} = \frac{N}{R}\left(\frac{\sum_\sigma m_1 e^{-H_N}\sum_{l=1}^R m_l}{Z} - \frac{\sum_\sigma m_1 e^{-H_N}\sum_\sigma e^{-H_N}\sum_{l=1}^R m_l}{Z^2}\right) \tag{11}$$

$$= \frac{N}{R}\Big(\langle m_1^2\rangle_{BG} + (R-1)\langle m_1 m_l\rangle_{BG} - R\langle m_1\rangle_{BG}^2\Big) \tag{12}$$

Since

$$J = \frac{1}{1-\mu^2} - \frac{1}{\chi} \tag{13}$$

its estimator will be:

$$\hat{J} = \frac{1}{1-\left(\frac{1}{R}\sum_{l=1}^R m_l\right)^2} - \frac{1}{\frac{N}{R^2}\sum_{l,s} m_l m_s - N\left(\frac{1}{R}\sum_{l=1}^R m_l\right)^2} \tag{14}$$

and

$$h = \tanh^{-1}(\mu) - J\mu \tag{15}$$

and his estimator is

$$\hat{h} = \tanh^{-1}\left(\frac{1}{R}\sum_{l=1}^R m_l\right) - \frac{\hat{J}}{R}\sum_{l=1}^R m_l \tag{16}$$

the old case instead said

$$\hat{J} = \frac{1}{1-\left(\frac{1}{M}\sum_{l=1}^M m(\sigma^m)\right)^2} - \frac{1}{\frac{N}{M}\sum_{l=1}^M m^2(\sigma^m) - N\left(\frac{1}{M}\sum_{l=1}^M m(\sigma^m)\right)^2} \tag{17}$$

$$\hat{h} = \tanh^{-1}\left(\frac{1}{M}\sum_{l=1}^M m(\sigma^m)\right) - \frac{\hat{J}}{M}\sum_{l=1}^M m(\sigma^m) \tag{18}$$

The method just identified will be soon applied to real data.