Scientific Report: Workshop on two-dimensional Diffusion Limited Aggregation

Hugo Duminil-Copin

Date. 02-07 december 2012

Place. Les Diablerets, SWITZERLAND

Organizer. Hugo Duminil-Copin

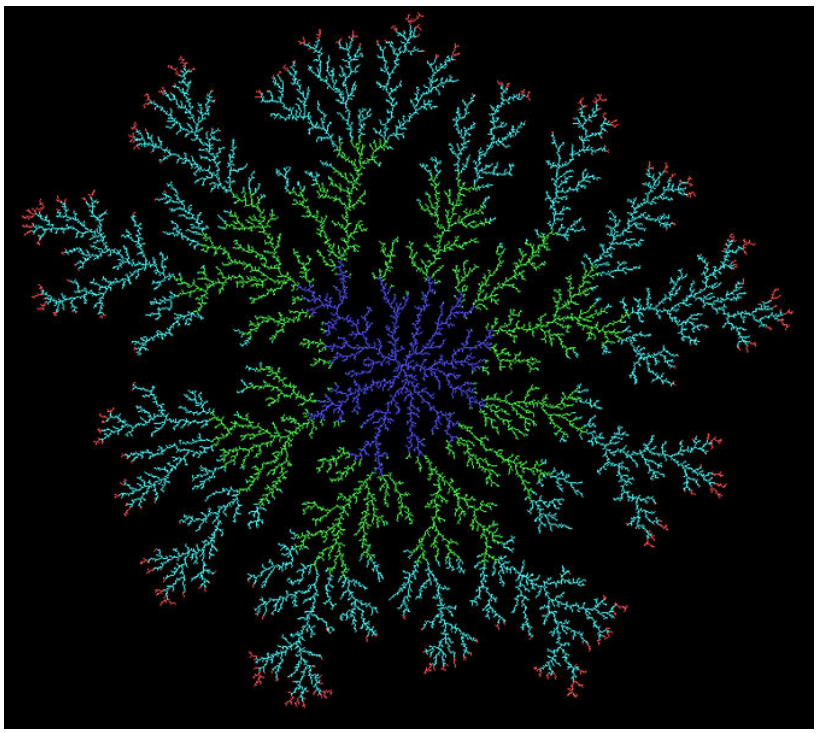
1 Summary

This workshop was intended to gather a small group of world experts on two-dimensional statistical physics and complex harmonic analysis in order to discuss the two-dimensional Diffusion Limited Aggregation (DLA). DLA is a model of aggregation for diffusive particles which produces a random fractal. It was introduced by Witten and Sander in 1981 and has attracted a lot of attention from both mathematicians and physicists. On $\mathbb{Z}^2$, an aggregate is constructed recursively by sampling a random walker "coming from infinity" and stopping it when hitting the existing aggregate. The penultimate vertex visited by the walker is then added to the aggregate.

While experiments allow to predict the behavior of the model quite efficiently, very little is known at the mathematical level. In fact, one of the only rigorous result was provided by Kesten, who proved that the radius of the aggregate built by n particles is $O(n^{2/3})$ with high probability. Attempts to approach the problem of DLA by slightly changing the model have been partially fruitful in the past decades. Benjamini and Yadin studied the DLA on cylinders. Makarov and Carleson used Loewner's equation to study a deterministic variant of DLA (which unfortunately is not fractal). Hastings and Levitov used conformal mapping techniques in order to investigate a more realistic version of DLA, albeit in a non-rigorous manner. It is fair to say that most of the models analyzed rigorously are very simplified versions of DLA and Kesten's result remains, after more than twenty years, the only rigorous result on two-dimensional Diffusion Limited Aggregation.

Recently, exceptional progresses have been achieved in planar statistical physics with the introduction of the Schramm-Loewner Evolution and the use of conformal invariance in the treatment of critical lattice models. The interaction between statistical physics and conformal invariance has thus been proved to be useful in several models and it seems therefore likely that this fecund connection can be used efficiently in the case of planar DLA. For this reason, we planned to gather specialists from planar statistical physics and complex harmonic analysis to proceed forward in the understanding of DLA.

The size of the workshop have voluntarily been kept small in order to favor interaction between all the participants. The participants of the workshop have been chosen according to their achievements in both fields and the intimate relation between their work and the DLA model. Long talks by each participant were used as basis for longer discussions on topics related to DLA. These discussions constituted the heart of the workshop.



2 Scientific content and discussions

The workshop was divided in talks followed by long mathematical discussions.

Talk of Steffen Rohde The workshop started by a 90 minutes talk by Steffen Rohde. He presented the Hastings-Levitov model and his joint work with Zinsmeister which provides a rigorous proof of the existence of a scaling limit for the model with parameter $\alpha = 0$ (the DLA corresponds to $\alpha = 2$). He mentioned the conjecture yielding that $\alpha = 1$ should be a critical point between fractal behavior and smooth behavior. The speaker also mentioned an idea discussed with Oded Schramm about uniformly penalizing very smooth configurations of the DLA in order to show that they were unlikely to occur in the model, thus proving some non-trivial upper bound on the speed of the DLA process (meaning the average number of particles needed to reach distance n). This observation was at the heart of the discussion that followed. It became quickly clear that this approach was not entirely sufficient to prove a non-trivial upper bound. The discussion ended by a short presentation of the work of Carleson and Makarov which further illustrates the connection between DLA and conformal mapping.

Talk of Stanislav Smirnov Stanislav Smirnov presented a strategy to tackle the following problem: Try to prove that there exists $\varepsilon > 0$ such that the average number of particles necessary to go to distance n is exceeding $n^{3/2+\varepsilon}$ (Kesten argument gives that this number exceeds $n^{3/2}$) by using a renormalization approach coupled with the fact that configurations corresponding to an aggregate growing at speed $O(n^{3/2})$ would be made of spikes which would not be stable configurations.

The participants started a discussion based on these ideas. The main goal of this discussion was to formulate a clear statement of the "renormalization step". Few possibilities were found, each one raising very difficult issues that were not solved during the workshop. Nevertheless, we believe that this approach could lead to further developments, and that this problem could eventually be solved. Note that we do not aim at a precise rate of growth, but only at an upper bound which would take the fractal nature of DLA into account.

Talk of Dmitry Beliaev Dmitry Beliaev presented his joint project with Stanislav Smirnov regarding non-trivial lower bounds for DLA growth. He introduced a toy model for which it is possible to prove that the density of particles is ultimately zero (this corresponds to showing that the average number of particles required to reach distance n divided by n^2 goes to infinity). The toy model is based on a simplification regarding the effect of adding one particle to the existing aggregate. Several generalizations were discussed, but tackling DLA along these lines seem still far from easy.

Talk of Gregory Lawler Greg Lawler proposed to look at the following quantity in terms of n. For the aggregate $A \subset \mathbb{Z}^d$ with $d \geq 3$, set

$$M_n = \sum_{y \in A} H_A(x)^3,$$

where $H_A(x)$ denotes the escape probability at the point x for the aggregate A. A similar definition can be made in $\mathbb{Z}^2$. The asymptotic behavior of this quantity can be controlled and it provides us with a rough estimate on the size of the DLA process. If this asymptotic behavior could be understood in more details, this could lead to better estimates on the rate of growth of the DLA process. Connections with the multi-spectral analysis of the aggregate, which boils down to the analysis of $\sum_{y \in A} H_A(x)^\alpha$ for every $\alpha > 0$, were also mentioned. Needless to say, multi-spectral analysis is very hard to perform on sets for which so little is known.

Talk of Itai Benjamini Itai Benjamini described his joint work with Ariel Yadin on DLA on cylinders $G \times \mathbb{Z}$ where G is an arbitrary finite graph. They prove that for sufficiently large graphs G (meaning that the simple

random walk on them mix sufficiently fast), the DLA indeed grows arms, or alternatively, that the expected time to reach distance n is much smaller than n^2. The result holds for expanders, tree-like graphs, but also much smaller graphs G. Unfortunately, the proof does not apply to $G = [-n, n]^d$, even for large d.

While extending the result to $G = [-n, n]^d$ seems as difficult has providing a non-trivial upper bound for the growth-speed of DLA, this result motivated the following natural question: consider the graph $[-n, n] \times \mathbb{N}$ and consider random walk coming from infinity and reflected on the boundary of this graph. Construct the DLA process recursively by starting with one particle at $(0, 0)$. What is the behavior, as n tends to infinity, of the expected time needed to reach the boundary $\{-n, n\} \times \mathbb{N}$.

This problem was discussed extensively during this workshop. Getting exponential bounds seems tractable (several possible strategies were proposed), and improving these bounds to polynomial represents a very interesting challenge. We wish to highlight that this question is closely related to the question of fjords in DLA, and whether they close eventually or not (this question is still controversial today, even in the physics literature).

Talk of Gady Kozma Gady Kozma presented his work with Gidi Amir, Omer Angel and Itai Benjamini investigating one dimensional DLA with long range (meaning that the law for a jump of random walkers is a priori supported on all $\mathbb{Z}$ and not on nearest neighbors). Even though the model is one-dimensional, interesting fractal behaviors emerge and power-law growths can be exhibited for relevant choices of jump laws. Still, the model is not understood in full generality and possible projects were discussed.

Talk of Vladas Sidoravicius Vladas Sidoravicius presented an alternative model for which particles are started according to a Poisson Point Process of small intensity on $\mathbb{Z}^2$ and perform random walks, or alternatively exclusion processes, before gluing to the existing aggregate (that started from the particle at the origin). The main question is whether the speed of growth of this process is linear.

Talk of Ronen Eldan Ronen Eldan presented a problem, proposed first by Itai Benjamini, related to the question of fjords in DLA. Without entering into unnecessary details, the model consists in an aggregate obtained by the aggregation of a shower of particles performing independent random walks on $\mathbb{Z} \times \mathbb{N}$ (more precisely, the initial aggregate is $\mathbb{Z} \times \{0\}$ and the random walks are starting well above the aggregate according to a Poisson Point Process and perform independent random walks until they hit

the existing aggregate). We keep a structure of tree by adding the bond corresponding to the last jump of a random walk before it hits the existing aggregate. The model is invariant under translation and one can ask whether trees survive forever or not. Good progress have been made on this problem during the workshop and it seems plausible that this question could be solve in the near future.

Talk of Vincent Beffara Vincent Beffara presented simulations and a joint work with one of his student dealing with directed DLA on $\mathbb{Z}^2$. Certain aspects of directed DLA are expected to be very different from the one of DLA. For instance, fluctuations should lie in the KPZ universality class. Nevertheless, long fjords are also created in this model and this phenomenon was compared to problems on DLA discussed above.

Talk of Hugo Duminil-Copin Hugo Duminil-Copin gave a summary of the work done during the workshop, sketching the different attempts and the specific problems that were discussed. This talk concluded the workshop.

3 Impact on the field

DLA being a very difficult problem, it will likely remain mysterious for a long time, but it is crucial to gather and unit specialists to try to proceed forward in its understanding (this is true for any difficult problem). The format of this workshop is perfectly suitable for this and should be encouraged in the future. As such, the workshop can represent a milestone in the development of the field.

Furthermore, it is necessary to regularly summarize the past research and revitalize the subject. This workshop fulfilled these two conditions. Several researchers started collaborating on several questions mentioned in the workshop, and their collaboration should carry on in the future. We hope and expect that ideas and strategies exchanged during this workshop will lead to further developments, and that the problems discussed in Benjamini's and Eldan's talks can be solved at least partially.

In conclusion, the workshop achieved his main goal: gathering world leaders to try to share ideas and partial advancements on this problem. Overall, talks and discussions were very beneficial for all the participants. As an illustration of the sharing spirit and chemistry between participants, informal evening sessions were also organized to carry on discussions started during the day.

4 Final program

Monday 03rd

9:00 - 10:30 Talk Steffen Rohde.
11:00 - 12:30 Discussion about Rohde's talk.
14:00 - 15:30 Talk Stanislav Smirnov.
16:00 - 17:30 Discussion about Smirnov's talk.

Tuesday 04th

9:00 - 10:30 Discussion about Smirnov's talk.
11:00 - 12:30 Talk Dmitry Beliaev.
14:00 - 15:30 Discussion about Beliaev's talk.
16:00 - 17:30 Discussion about Beliaev's talk.

Wednesday 05th

9:00 - 10:30 Talk Greg Lawler.
11:00 - 12:30 Discussion about Lawler's talk.
14:00 - 15:30 Talk Itai Benjamini.
16:00 - 17:30 Discussion about Benjamini's talk.

Thursday 06th

9:00 - 10:30 Talk Gady Kozma.
11:00 - 12:30 Talk Vladas Sidoravicius.
14:00 - 15:30 Talk Ronen Eldan.
16:00 - 17:30 Discussion about Eldan's talk.

Friday 07th

9:00 - 10:30 Talk Vincent Beffara.
11:00 - 12:30 Talk Hugo Duminil-Copin.