EUROPEAN SCIENCE FOUNDATION
SETTING SCIENCE AGENDAS FOR EUROPE

RESEARCH NETWORKING PROGRAMME

GLOBAL AND GEOMETRIC ASPECTS OF NONLINEAR DIFFERENTIAL EQUATIONS (GLOBAL)

Standing Committee for Physical and Engineering Sciences (PESC)

The European Science Foundation (ESF) was established in 1974 to create a common European platform for cross-border cooperation in all aspects of scientific research.
With its emphasis on a multidisciplinary and pan-European approach, the Foundation provides the leadership necessary to open new frontiers in European science.
Its activities include providing science policy advice (Science Strategy); stimulating co-operation between researchers and organisations to explore new directions (Science Synergy); and the administration of externally funded programmes (Science Management). These take place in the following areas:
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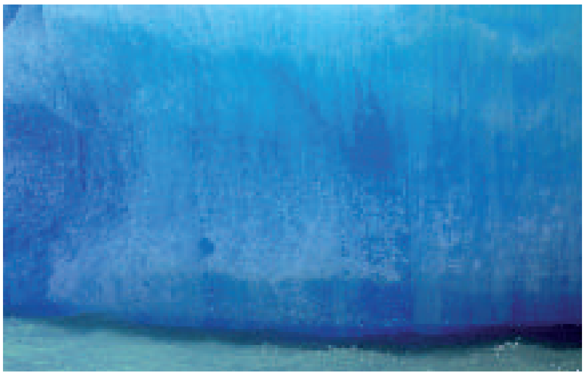
Headquartered in Strasbourg with offices in Brussels, the ESF's membership comprises 75 national funding agencies, research performing organisations and academies from 30 European countries.
The Foundation's independence allows the ESF to objectively represent the priorities of all these members.

Cover picture:
Rice Paddies: Diffusion and Convection in Heterogeneous Environments

Introduction



Free Boundaries on a Melting Iceberg

Nonlinear partial differential equations is a field where pure and applied mathematics come together most beautifully. Encouraged by the many applications in physics, chemistry, biology, medicine and finance many groups of researchers all over the world and particularly in Europe have specialised in the study of nonlinear partial differential equations. It has become an independent field with various research directions. New phenomena must be explained, such as blow-up in nonlinear heat equations, compact support questions for equations of porous media, free boundaries in obstacle problems. Their mathematical discussion requires new theoretical and numerical tools.

This programme aims at the study of global and geometric properties of solutions of nonlinear partial differential equations from an analytical point of view. It focuses on particular problems such as nonlinear evolution, optimal transportation problems, free boundary problems, nonlinear diffusive systems, singular perturbations, nonlinear stochastic partial differential equations, regularity issues and global solutions.

The subject matter has a wide scope, but nevertheless there is a common language which should be explored and used more systematically.

The strategy is to build a network among active groups and individual scientists in Europe.

Principal Goals

- Solutions of open problems.
- Formulation of new problems and opening up of new research directions.
- Stimulation of interdisciplinary contacts.
- Training of students and junior researchers.

The running period of the ESF GLOBAL Research Networking Programme is for five years from June 2004 to June 2009.

Main Topics

Optimal Transportation Problems

The problem of optimal transportation of mass has been much in focus in recent years. Its analysis has revealed new applications of nonlinear partial differential equations in fluid mechanics, shape optimisation, geometric and functional inequalities, dynamical systems, Aubry-Mather theory and viscosity solutions of Hamilton-Jacobi equations, statistical mechanics, probability, economics. The original problem stated by Monge in 1781 and later generalised by Kantorovich in 1942 reads as follows:

> *Given is a pile of soil or rubble with a given mass density that has to be transported to an excavation with another mass density. Under the assumptions that the total masses are equal, find a transport scheme that carries the first distribution into the second and minimises the transportation cost.*

This problem can be generalised in many ways, entailing, for example, more general distribution functions of equal mass in more general spaces and various cost functions. Its solution involves techniques of ordinary and partial differential equations. The geometric and qualitative aspects of the problem such as existence, uniqueness and characterisation of the optimal transport have been studied by many authors.

Because of its practical importance and its inherent mathematical beauty in the last few years there have been several workshops/conferences devoted to this subject and more will follow.

Monge Mass Transportation: A ‘Deblais’

Nonlinear Diffusion Equations

Nonlinear parabolic equations arise in areas such as fluid filtration, mass diffusion, particle transport and heat propagation. More recently, work has been orientated to new fields of applications such as image processing. Degeneracy or singularities of these equations often leads to free boundaries. Emphasis will be placed on the following topics.

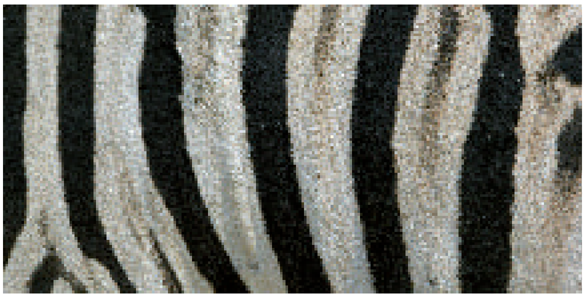
Main Topics

1. *Self-similarity and asymptotic methods*. These methods are crucial when dealing with solutions which are invariant under transformation groups, a classical subject in continuum mechanics. Moreover we study properties of stability and domains of attraction, cf. topic 3 below.
2. *Blow-up and extinction*. These questions come up in combustion theory. Two questions are of main interest: i) the explanation and calculation of phenomena of complete blow-up; ii) the theory of blow-up for systems.
3. *Higher-order equations*. The basic model is the thin film equation. New applications for this equation can be found in quantum mechanics and nanotechnology. Because of the lack of comparison principles an analysis of the related variational inequalities, of obstacle problems and of singularity formation is challenging.
4. *Problems with strong convection*. Equations of this type are used, for example, in models of multiple transitions of phase (water-ice-vapour), in biomathematics (chemotaxis), in the flow of a fluid through a porous medium and in semiconductor devices. Questions related to uniqueness, regularity and asymptotic behaviour including (uniform) boundedness and Harnack estimates are still open.

Entropy Methods for Nonlinear Diffusion and Kinetic Equations

In recent years a new technique has been developed for the analysis of the long-time behaviour of diffusive systems. This technique, now known as the 'entropy dissipation method', is based on Boltzmann´s original work in gas kinetics. The method is based on finding a time-decaying functional of the solution of the diffusive system under consideration, satisfying appropriate functional properties (such as convexity) and deriving an equation for it in terms of its dissipation. In particular, the time-decay of (relative) entropy can be determined when an estimate of the entropy dissipation in terms of (relative) entropy is known. Recently, this technique has led to exciting new results for Boltzmann equations, the Landau equation, quantum kinetic models, and linear and nonlinear Fokker-Planck type systems. An important spin-off has been the proof of new Sobolev-type inequalities. The aim is to apply the entropy-entropy dissipation technique to more complicated partial differential equations, such as inhomogeneous (classical and quantum) kinetic systems, energy transport equations, and Fokker-Planck systems with non-conservative drift.

Ridge of a Sand Dune: Granular Material Flow



Pattern Formation on Zebra Skin: Turing Instability

Singular Perturbation and Homogenisation of Nonlinear Partial Differential Equations

Many physical systems can be modelled as nonlinear partial differential equations depending on a small parameter, which multiplies a highest order derivative. In general, the solutions of these equations exhibit singular behaviour as this parameter tends to zero. Prob lems of this type, which gained significant importance in Prandtl's fluid dynamics boundary layer theory, are termed 'singular perturbation problems'. Very often, an efficient numerical simulation of these problems and a qualitative (physical) understanding of these equations is impossible without deep mathematical analysis and qualitative information on solutions. Typically, singular perturbation of nonlinear partial differential equations leads to complicated boundary layer behaviour, to the occurrence of free boundaries and to singularities in physical space and/or time. Moreover, singular perturbation of kinetic equations gives rise to diffusive equations in position space. The following topics will be investigated:
1. Singular perturbations in flame propagation and in systems with dominant convection.
2. Small Debye-length limits in semiconductor energy – transport and drift diffusion models.
3. Mean-free-path limits of kinetic equations. Homogenisation problems are characterised by small wavelength oscillations of the coefficients of the differential operators. The difficulty is to find approximate

simpler problems, the solutions of which exhibit the correct average (or macroscopic scale) behaviour without necessarily resolving the fine microscopic structure. Important homogenisation problems occur in geophysics, quantum transport in crystals and electromagnetic fields in periodic media. Progress in homogenisation problems of quantum physics is expected, connected to the already well-developed theory of homogenisation of Hamilton-Jacobi equations by means of viscosity solutions. Other problems to be treated include the homogenisation of the energy density of linear and weakly nonlinear double scale PDEs by means of Wigner transforms. Also homogenisation problems in kinetic theory and kinetic equations in random media will be analysed. There are close connections to transport problems in random media. Therefore there is a link between this topic and the nonlinear stochastic partial differential equations mentioned below.

Regularity of Free Boundaries

In a physical system where, for example, different phases meet at certain interfaces, the location of the interfaces constitutes a free boundary that separates regions in which different differential equations hold. Regularity or smoothness questions relating to free boundaries have been dealt with for several decades. It is important to analyse these boundaries from a rigorous mathematical point of view in order to understand their structure and to devise stable numerical algorithms for their computation.

Recent years have also seen many new applications requiring new techniques. Several areas such as potential theory, micromagnetics, image processing, computer vision, geometric measure theory, free boundary problems in physics/mechanics and many other subjects have profited from the new techniques.

The subject is growing at a fast pace and experts predict the resolution of several important unanswered questions within the next 5-10 years. A famous problem of this kind is the regularity of the free boundary for a minimisation problem, which leads to the so-called 'Bernoulli-type' free boundary condition.

Global Solutions in Partial Differential Equations and Free Boundary Problems

The local study of solutions of partial differential equations is often carried out through a technique of blow-up analysis (or zooming). This procedure reduces the problem to the classification of global solutions in the whole space. It is employed, for instance, for minimal surfaces, free boundary problems and for semilinear equations.

An important question in this direction is a conjecture of De Giorgi from 1978. The problem concerns the symmetry of bounded entire solutions of semilinear elliptic equations. The global problem is obtained after blowing up an interface or phase transition between two different physical or chemical states, and it can be considered to be a perturbed version of the classification of entire minimal graphs. The conjecture is that their level sets are hyper planes.

Fully Nonlinear Partial Differential Equations

Fully nonlinear elliptic equations are nonlinear in their highest derivatives and have important applications in many areas. In global differential geometry a prototype is the Monge-Ampère equation. The study of global problems concerning hypersurfaces of positive constant Gauss curvature and the corresponding classification of global solutions are among the current interests. The connections between affine differential geometry and k-Hessian equations (equations involving principal curvatures) is also a topic under intense current investigation. A second field of great activity in fully nonlinear equations has arisen recently, following new results and applications of Monge-Ampère equations to problems of optimal transport maps (or allocation problems) and of optimal design.

Finally, the study of fully nonlinear equations in periodic media and their homogenisations, as well as connections with stochastic Aubry-Mather theory and eventually quantum mechanics are also being studied at present.

Nonlinear Stochastic Partial Differential Equations

In many real life situations in economics and elsewhere, stochastic effects must be taken into account when modelling with differential equations. For example, the theory of optimal stopping is central in economics. The problem of finding the optimal time of starting (or stopping) an economic activity under uncertainty can often be formulated as an optimal stopping problem. In fact, it has been shown that the price of an American option in finance is described by the solution of an optimal stopping problem. Today it is well known that an optimal stopping problem for a diffusion is equivalent to a free boundary problem, and such problems are again equivalent to certain variational inequalities involving partial differential operators.

Stochastic control problems are also central in eco-

Main Topics

nomics. Examples are the problem of finding the optimal consumption rate and the optimal portfolio for a trader in financial markets. The dynamic programming method gives a direct link between the problem of stochastic control of a diffusion and a nonlinear partial differential equation (the Hamilton-Jacobi-Bellman equation) which involves the generator of the diffusion.

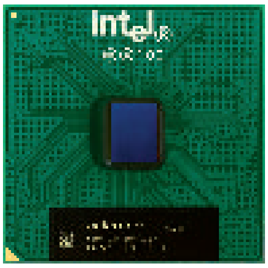
There are also stochastic partial differential equations in mathematical finance. The Musiela equation is an example that occurs in the theory of interest rates in finance.

Filtering theory is central to finding the best estimate at a given time of a stochastic system based on noisy observations of the system. In the linear case the famous Kalman filter solves this problem explicitly. In the nonlinear case the solution is described in terms of a stochastic partial differential equation, called the 'Zakai equation'.

Applications

The focus will be on specific models related to the filtration of fluids in porous media. In particular we investigate:

1. *Gravity segregation in porous media*. The aim is to study the segregation of foam in horizontal reservoirs. This leads to a three-phase elliptic free boundary problem with free boundaries separating the regions containing foam, gas and water. In particular, the behaviour of the three free boundaries near the triple point (where complete segregation takes place) poses new and challenging questions.
2. *Crystal dissolution and precipitation*. This involves a model for the convective-dispersive transport of solutes in a porous medium undergoing precipitation/dissolution reactions with respect to the solid matrix. In mathematical terms one needs to consider two transport equations and a set-valued first order reaction equation. A preliminary travelling wave analysis indicates the conditions for which dissolution/precipitation fronts (free boundaries) occur. Mathematical issues are the regularity and qualitative properties of the free boundaries and the stabilisation of profiles towards travelling waves.



Semiconductors:
Intel Celeron Processor

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For the latest information on this Research Networking Programme consult the GLOBAL website:
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