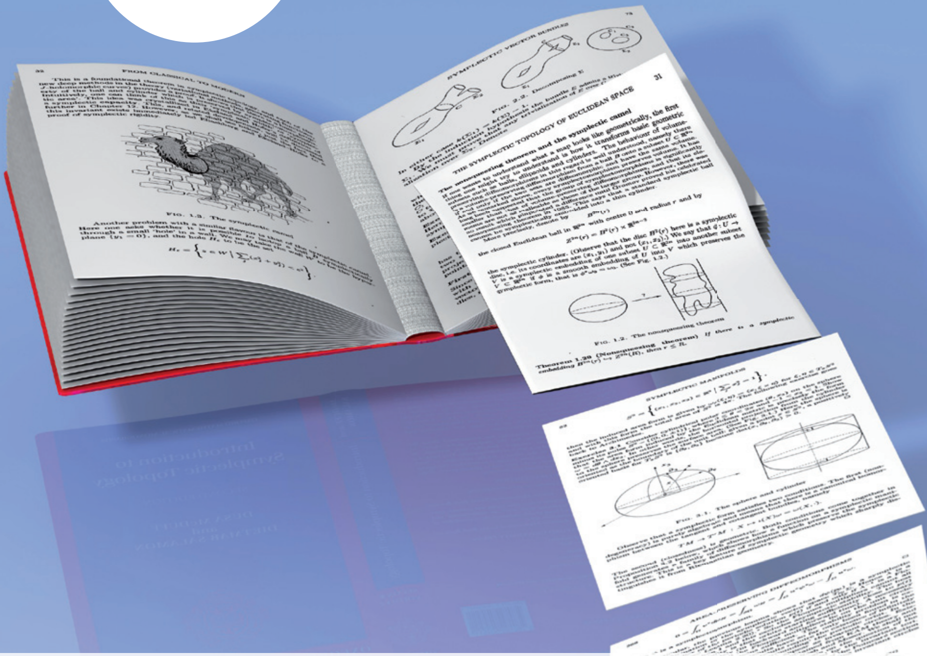
EUROPEAN SCIENCE FOUNDATION



Research Networking Programme

Contact and Symplectic Topology (CAST)

Standing Committee for Physical and Engineering Sciences (PESC)

Symplectic and contact geometry are theories that naturally emerged from the mathematical description of classical physics. They were revolutionised in the early 1980s with the discovery of new rigidity phenomena and properties satisfied by these geometric structures. Since then they have been very useful in the development of many areas of mathematics and modern mathematical physics.

In recent years a large number of young researchers in contact and symplectic geometry established active research groups in their home countries. Thus Europe has become a stronghold of the subject. This is reflected by an increase in workshops and conferences held in Europe.

The goal of the CAST Research Networking Programme is to stimulate exchange between researchers from all branches of contact and symplectic topology, in order to create a comprehensive perspective on the field and make progress on some of the basic open questions.

The European scale of this network, involving research teams from 13 different countries, reflects the global nature of these questions as well as the European strength in the subject. The planned activities include workshops, research collaborations, and the exchange of doctoral students and postdoctoral researchers.

More information about CAST activities can be found at:
http://cast.ulb.ac.be

The running period of the ESF CAST Research Networking Programme is for five years from January 2010 to January 2015.

Scientific Background

Contact and symplectic structures have their origin in classical physics. Symplectic manifolds are the natural phase spaces for classical mechanics in its Hamiltonian formulation. Contact manifolds are the natural framework for geometric optics. More generally, contact structures arise naturally on energy levels of autonomous Hamiltonian systems. Many basic concepts such as symplectic reduction, Hamilton–Jacobi equations, Legendrian and Lagrangian submanifolds were developed in this context and continue to play an important role to this day.

Symplectic structures appear in even dimensions, while contact structures live in odd dimensions. There is an extremely rich interplay between these two types of geometric structures, so that similar techniques can be used to study both of them.

Important developments in the early 1980s led to the emergence of new questions about symplectic and contact structures. The field addressing these questions is called contact and symplectic topology. Its birth may be dated to the following events: the proof of the Arnold conjecture on fixed points of Hamiltonian diffeomorphisms for surfaces by Eliashberg (1979) and for the torus by Conley and Zehnder (1982), Bennequin's proof that the standard contact structure on R^3 is tight (1982), and Gromov's proof of his "non-squeezing theorem" (1985). Since then the subject has grown into one of the most active and vibrant areas of mathematical research, with profound contributions by many leading mathematicians.

Since then, the most spectacular achievements in this recent field include:

- Gromov's theory of pseudo-holomorphic curves in symplectic manifolds (1985);
- the development by Chaperon, Laudenbach and Sikorav of the theory of generating functions with applications to Lagrangian intersections (1984-1987);
- Eliashberg's classification of overtwisted contact structures (1989);
- Floer's work on the Arnold conjecture and the introduction of Floer homology (1989);
- Hofer's metric on the group of Hamiltonian diffeomorphisms via Hamiltonian dynamics (1990);
- Viterbo's distance on the group of Hamiltonian diffeomorphism via generating functions (1992);
- the Fukaya category and its relation to mirror symmetry (1993);
- spectacular results on symplectic packing by McDuff and Polterovich (1994), followed by many other authors;
- Kontsevich's introduction of the concept of a "stable map" and the subsequent emergence of Gromov-Witten theory (1995);
- Donaldson's theory of asymptotically holomorphic sections and Lefschetz pencils (1996);
- Taubes' correspondence between solutions of Seiberg–Witten equations and pseudo-holomorphic curves (1996);
- symplectic field theory by Eliashberg, Givental and Hofer (2000);
- Giroux's open book decompositions for contact structures (2002);
- Legendrian contact homology by Chekanov, Ekholm, Etnyre and Sullivan (2002);
- the construction of quasi-morphisms on groups of Hamiltonian diffeomorphisms by Polterovich and Entov (2003) and their first application to rigidity of intersection by Biran, Polterovich and Entov (2004);
- the discovery of contact non-squeezing by Eliashberg, Kim and Polterovich (2004);
- Heegaard Floer homology by Ozsváth and Szabó (2004);
- the proof of Arnold's Four Cusps conjecture by Chekanov and Pushkar (2005);

- the discovery of exotic Stein structures on C^n by Seidel, Smith and McLean (2007);
- Taubes' proof of the Weinstein conjecture in dimension 3 (2007).

A distinguishing feature of contact and symplectic topology has always been its close interactions with other fields: complex and algebraic geometry, low-dimensional topology, and mathematical physics (classical mechanics, quantum field theories, gauge theories, string theory).

While a lot of progress has been made on some of the big problems in the field (Arnold conjecture, Weinstein conjecture in dimension 3, contact structures and Legendrian knots in dimension 3), many of the most basic questions are still wide open: existence of contact and symplectic structures above dimension 4, the structure of symplectomorphism and contactomorphism groups, and dynamical results beyond the existence of periodic orbits. Moreover, the methods in the field have become increasingly sophisticated and are sometimes not easily accessible (e.g. Fukaya categories and symplectic field theory).

Research Topics

Fukaya categories and mirror symmetry

Fukaya categories package a great deal of information about a symplectic manifold and its Lagrangian submanifolds. However, it is rarely obvious how to extract that information, so they have yet to become a widely used tool in symplectic topology. Typical applications so far include restrictions on the topology and intersection properties of certain classes of Lagrangian embeddings in some symplectic manifolds. There is scope for progress on a wide range of questions, relating directly to other research topics. This includes the study of exotic symplectic structures and of the fixed points for some classes of symplectomorphisms.

Mirror symmetry establishes a very strong relation between holomorphic curve invariants of a symplectic manifold X and sheaf theoretic quantities associated to a complex manifold Y, called the mirror of X.

It can be used to make predictions for the number of holomorphic curves in some symplectic manifolds. Such predictions have been proved rigorously in a number of cases. Mirror symmetry will be an important tool for the computation of the Fukaya category.

Important questions involving Fukaya categories are directed towards other research topics. These include relations to Hamiltonian dynamics and entropy, the relation via Hochschild homology to symplectic homology, and the search for generators of Fukaya categories in low-dimensional topology.

Floer homology and Hamiltonian dynamics

Hamiltonian dynamics is at the interface of classical mechanics and symplectic geometry. This flourishing topic addresses important questions emanating from the theory of dynamical systems. The

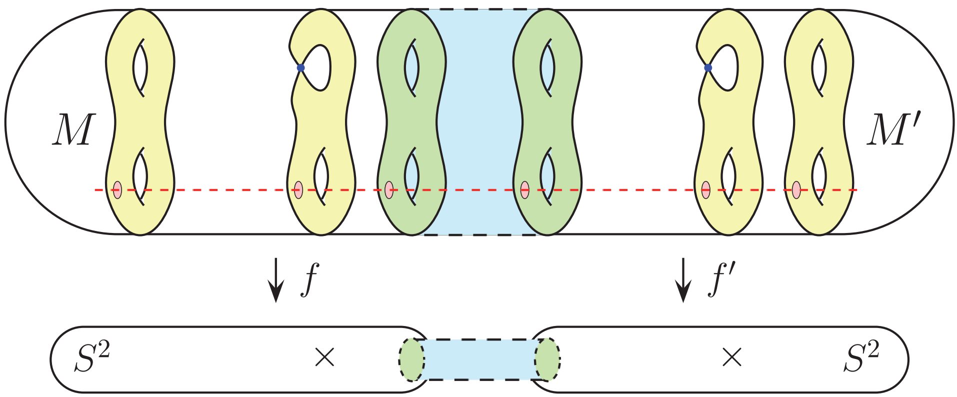


Figure 1. Lefschetz fibrations can also be used to compute Fukaya categories.

Arnold conjecture gives a topological lower bound on the number of periodic trajectories with a fixed period. The Weinstein conjecture states in particular that every convex energy level (i.e. naturally equipped with a contact structure) carries at least one periodic trajectory. These major conjectures have been a driving force in the development of methods and theories in the context of symplectic rigidity phenomena, most notably Floer homology and its contact geometrical analogues: contact homology and, more generally, symplectic field theory. Although the Arnold conjecture and the 3-dimensional Weinstein conjecture are settled by now, the methods developed in this course have been extended further and directed towards a broad spectrum of new applications.

Important questions in this vein include more precise quantitative results for the number of periodic trajectories, the study of packing or displacement problems, the understanding of the nature of the holomorphic curves invariants developed, and a deeper understanding of Hamiltonian systems of physical type.

Symplectic field theory

Symplectic field theory (SFT) stands for a program to create a unified theory of holomorphic curves in symplectic manifolds. Building on the seminal work of Gromov and Floer, this program was first formulated by Eliashberg, Givental and Hofer in 2000. In its ultimate form, SFT will incorporate all existing holomorphic curve theories such as Gromov–Witten theory, Floer homology, Legendrian contact homology, and the Fukaya category. Applications of SFT cover such diverse areas as Hamiltonian dynamics, symplectic and contact topology, smooth topology, complex and algebraic geometry, and fluid dynamics.

The most exciting questions about SFT include: the computation of SFT for Stein manifolds, the application of SFT to the study of smooth structures via conormal bundles, the development of a relative version of SFT for Legendrian submanifolds, the relation between SFT and the Fukaya category, the SFT interpretation of the contact class in Heegaard Floer homology, the vanishing properties of SFT, the obstructions obtained from SFT to fillability of contact manifolds and Legendrian knots and to Lagrangian embeddings or immersions.

Contact topology

Contact topology is now the meeting point of many research areas: low dimensional topology, foliation theory, dynamical systems, symplectic topology, and Floer homology theories. Great progress was recently made in this topic, thanks to Giroux's description of contact structures in terms of open book decompositions.

In dimension 3, there is a fundamental dichotomy between tight and overtwisted contact structures. Overtwisted contact structures were classified topologically by Eliashberg. Tight contact structures were geometrically classified on important classes of 3-manifolds; they also admit a coarse classification for all compact 3-manifolds. Several fundamental questions that are still unanswered include: the preservation of the tightness property under Legendrian surgery, the existence of a tight contact structure on any irreducible 3-manifold, the classification of tight contact structures on closed Seifert fibered 3-manifolds, the classification of Legendrian and transverse knots in fixed knot types, the classification of symplectic fillings of links of isolated surface singularities.

In higher dimensions, the topological characterisation of closed manifolds which admit contact structures is a fundamental problem which is completely open. More precisely, it is still unknown whether any almost contact manifold admits a contact structure. The description of a contact structure in terms of an open book decomposition by Giroux and Mohsen provides a possible strategy to answer this question. This description has already been used by Bourgeois to show that every odd-dimensional torus carries a contact structure.

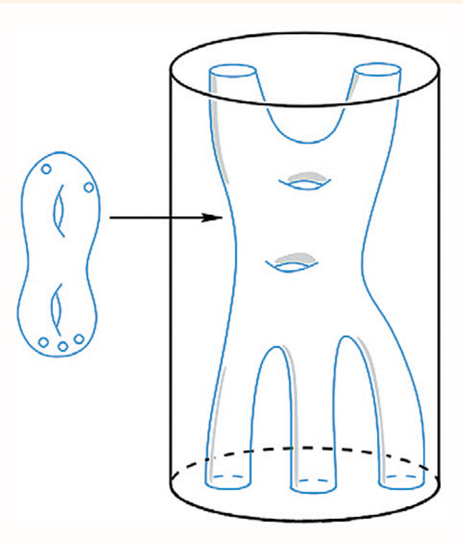


Figure 2. Holomorphic curve considered in SFT.

Complex geometry and Stein manifolds

Complex geometry naturally provides distinguished classes of symplectic manifolds, such as Kähler manifolds and Stein manifolds. Stein manifolds have played a central role in the development of complex and algebraic geometry in the 20th century, with deep contributions by H. Cartan, Grauert, Hörmander and many others. In 1991 Eliashberg provided an entirely new view on the subject by giving a purely topological characterisation of all manifolds of even dimension at least 6

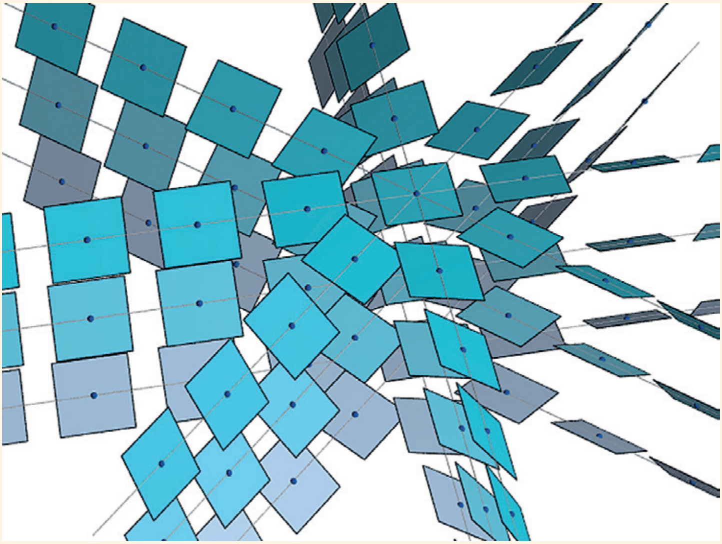


Figure 3.
The standard contact structure on R^3.

admitting a Stein structure. His proof relies on a range of techniques from differential, symplectic and contact topology. The complexity of Stein structures lies in the critical cells (of half the dimension of the Stein manifold) and recent developments concentrate on their study. Important questions about Stein manifolds include the study of non homotopic Stein structures on a given manifold, in order to produce exotic symplectic structures, and possibly exotic contact structures on the boundary.

Other topics from complex geometry are the existence of Kähler metrics of constant scalar curvature on complex projective varieties, and Donaldson–Thomas theory, which can be viewed as a complexification of Floer theory.

Topology of symplectic manifolds

Closed Kähler manifolds have many special homotopical properties (e.g. even odd-dimensional Betti numbers, hard Lefschetz property, vanishing Massey products, formality). On the other hand, recent developments have shown that all homotopic restrictions on Kähler manifolds are violated by symplectic manifolds. This adds evidence to the conjecture due to Thurston that symplectic manifolds have no distinguished cohomological properties.

An important theme in symplectic topology is the relation between the group of symplectic diffeomorphisms and the whole diffeomorphism group. For instance, the study of the isotopy class for some symplectomorphisms of the standard symplectic torus is related to the question of existence for symplectic structures on exotic tori.

Another important question in symplectic topology is related to monotone Lagrangian submanifolds: is it true that their Floer homology either vanishes or is isomorphic to their singular homology?

Groups of symplectomorphisms and contactomorphisms

These groups of diffeomorphisms preserving symplectic or contact structures play a central role in contact and symplectic topology. For some classes of symplectic manifolds, the homotopy type of the symplectomorphism group is completely understood. Moreover, some information is known about their cohomology. A general question in this vein is to understand to what extent the topology of the symplectomorphism group is determined by compact subgroups arising from Lie group actions.

Symplectic quasi-states are functionals on the space of functions on a symplectic manifold, corresponding to states in quantum mechanics. These quasi-states can be constructed via quasi-morphisms on the groups of Hamiltonian symplectomorphisms or via symplectic homogenisation. Future research topics include the study of the connection between these two constructions, and the study of quasi-morphisms on groups of symplectic homeomorphisms.

The homotopy groups of the contactomorphism groups are related to those of the space of contact structures. The latter have been studied recently via 3-dimensional techniques, and via holomorphic curves. On the other hand, it was shown that contactomorphism groups either carry a partial order or contain a semigroup of positive contractible loops. This dichotomy should be explored for some classes of contact manifolds, and the semigroup should be computed in the nonorderable cases.

Workshops, schools and conferences

The CAST Research Networking Programme will support several continuing series of workshops in the field:

- workshops on symplectic geometry, contact geometry and Interactions, bringing together 60–70 researchers for exchange of recent results and ideas,
- workshops on symplectic field theory, consisting of lecture series on specific topics by international experts and preceded by a two-day precourse to prepare for the lectures,
- GESTA workshops, offering to students and young researchers a series of mini-courses on contact and symplectic topology.

In addition to the above workshop series, CAST will support other scientific meetings in the field of contact and symplectic topology. Proposals for science meetings to take place during the next calendar year should be submitted at **http://cast.ulb.ac.be**

Short visits and exchange grants

The CAST programme will finance visits to research teams abroad, in order to encourage the numerous existing collaborations between research teams throughout Europe. There are two types of visits:

- short research visits of up to 15 days,
- longer exchange visits, from 15 days to three months.

The applications for short visits should be received at least two weeks before the starting date. This delay is extended to one month for exchange grants. Applications will be reviewed on a running basis.

For all further information please see **http://cast.ulb.ac.be** or contact the network coordination at **cast@ulb.ac.be**

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For the latest information on this Research Networking Programme consult the CAST websites:
cast.ulb.ac.be
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Cover picture:
An artist's view of a contact structure supported by a symplectic open book.

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