European Science Foundation

SETTING SCIENCE AGENDAS FOR EUROPE

RESEARCH NETWORKING PROGRAMME

NEW FRONTIERS OF INFINITY: MATHEMATICAL, PHILOSOPHICAL AND COMPUTATIONAL PROSPECTS (INFTY)

Standing Committee for the Humanities (SCH)
Standing Committee for Physical and Engineering Sciences (PESC)

Introduction

The European Science Foundation (ESF) is an independent, non-governmental organisation, the members of which are 80 national funding agencies, research-performing agencies, academies and learned societies from 30 countries.
The strength of ESF lies in the influential membership and in its ability to bring together the different domains of European science in order to meet the challenges of the future.
Since its establishment in 1974, ESF, which has its headquarters in Strasbourg with offices in Brussels and Ostend, has assembled a host of organisations that span all disciplines of science, to create a common platform for cross-border cooperation in Europe.
ESF is dedicated to promote collaboration in scientific research, funding of research and science policy across Europe. Through its activities and instruments ESF has made major contributions to science in a global context. The ESF covers the following scientific domains:

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Cover picture:
Caspar David Friedrich (1774, Greifswald — 1840, Dresden).
Monk by the Sea (1809)
Oil on canvas, 110 × 172 cm
National Gallery, Berlin.

Human beings have been always fascinated by infinity. From ancient Greece, philosophers and mathematicians have studied its nature and been bewildered by the paradoxes that defy its rational comprehension. In medieval times, infinity was thought of as the ultimate attribute of God, and as such totally unassailable to humans. But the birth of modern physics in the 17th century was only possible thanks to the new mathematical ideas about the infinite introduced by Newton and Leibniz in their differential and integral calculus. Indeed, one of the principal reasons for the surprising success of mathematics in natural sciences, and even in humanities and social sciences, is its ability to use the concept of infinity to model real-world objects and phenomena. In the second half of the 19th century, deep investigations by Dedekind, Weierstrass and others about the nature of infinite mathematical objects and, above all, the dramatic discoveries of Cantor on transfinite numbers and his new theory of abstract sets — the first true investigation of the infinite realm — made it possible, and necessary, to establish a purely mathematical theory of infinity. Their investigations were decisive for the revolutionary conceptual transformations that took place at the turn of the 20th century, both in mathematics and in physics. Hilbert said in 1925: "the clarification of the nature of infinity has become necessary, not only for the special interests of the particular sciences, but mainly for the honour of human understanding". However, Hilbert's attempt to reduce the infinite to the finite in order to clarify its nature could not succeed, for Gödel's fundamental work on the incompleteness phenomena in the 1930s demonstrated that the use of infinitary methods in mathematics is not only a convenient tool, but an essential requirement for our understanding of even such basic notions as the natural numbers.

The study of infinity was firmly established as a mathematical discipline in the early 20th century by Zermelo, Fraenkel, Gödel, von Neumann and others, culminating in the current ZFC (Zermelo-Fraenkel with Choice) system of modern set theory. This discipline, currently known as set theory, has as its goal the elucidation of the nature of infinite mathematical objects and their role in foundational issues underlying mathematics. It links together deep research on hard mathematical problems involving infinite objects and philosophical investigation on the nature of rational thinking. Moreover, ZFC is the usual framework for mathematics. That means that all mathematical objects, both finite and infinite, can be construed as sets, and their basic properties can be logically derived from the ZFC axioms, which gives this theory a foundational status and endows it with philosophical significance. The unified framework provided by ZFC for regular mathematics is now also being developed for constructive, and even for computable, mathematics.

Aims and Objectives

Illustration from Camille Flammarion's book "L'Atmosphere: Météorologie Populaire", Paris, 1888.

Research on infinite sets and objects has produced remarkable results and unexpected developments. Most surprisingly, it made it possible to prove mathematically that some fundamental questions about the continuum — arguably the most important infinite mathematical object – such as Cantor's Continuum Hypothesis, Suslin's Hypothesis or the Measure Problem of Ulam, cannot be settled using the standard mathematical tools, which are embodied in the ZFC system. For this, new techniques were developed, such as Gödel's theory of *constructible sets*, Cohen's *forcing* method, and the theory of *large cardinals*. These are very powerful instruments which, as Gödel predicted, are intimately interconnected and have developed into the current mainstream set-theoretic areas of research, such as inner model theory, fusing constructibility and large cardinals, and generic absoluteness, linking large cardinals and forcing.

Set-theoretical research has been a continuous source of original ideas and results, as well as of deep and highly technical tools that are now finding applications in many areas of mathematics, computation, and even natural and social sciences.

The main objective of the Programme is to stimulate the exchange of ideas among researchers pursuing different approaches to infinity: mathematical, philosophical and computational. Its aim is to promote cooperation at European and international levels, scientific mobility and integration of national activities and groups with complementary backgrounds and expertise, and training of young researchers.

The ESF Research Networking Programme INFTY will run for a period of five years, from March 2009 to March 2014.

The main general objectives are (i) to create an interdisciplinary forum for sharing knowledge and expertise, and stimulating the exchange of ideas among researchers working on mathematical, philosophical and computational aspects of infinity, and (ii) to coordinate the work of different groups from different areas and countries, and training of young researchers, with special attention to women and researchers from economically less-developed countries, in an integrated European multidisciplinary environment with infinity as the common theme.

INFTY envisages:

a) Facilitating, integrating and disseminating the work carried out by the set-theoretic communities in different countries of Europe;
b) Stimulating interdisciplinarity, by promoting the coordination and exchange of views between leading groups in different areas and disseminating their shared visions of the new frontiers to the whole community;
c) Promoting the fast transmission of new concepts and techniques from the research frontiers to the basic training level.

The expected benefits from the Research Networking Programme include (i) a better understanding, and possible solution, of some fundamental problems about infinite sets through cross-fertilisation of different approaches, mainly mathematical and computational, together with a philosophical analysis of the ideas involved; (ii) development of new techniques originated in set theory to deal with infinite entities to other areas of mathematics and science; (iii) identification of new application areas of set theory and the new technical and conceptual advances required; (iv) consolidation of INFTY as the common meeting ground for research on infinity in Europe; and (v) enhancing the visibility and attractiveness of European set theory for scientists and students in third countries.

INFTY involves teams and individual researchers from 10 European countries. All teams are well-established and led by renowned researchers with a high international profile. ***The Programme includes all leading research groups in set theory pertaining to ESF countries***.

About 60% of the initial INFTY members are set theorists. Many of them work also on related mathematical areas and have interest in philosophical, foundational and computational issues involving infinity. 25% of the members work on areas of mathematics that apply set-theoretic methods, such as infinite combinatorics, Boolean algebras, or Banach space theory. 7% are philosophers, working on philosophy of set theory and of general mathematics, and 11% are researchers on theoretical computer science, constructive mathematics and computational linguistics.

Aims and Objectives

The national teams have complementary unique techniques, which will be jointly used to achieve the objectives stated above. This collaboration will be further enhanced by complementary techniques and expertise of visiting partners from Israel, Japan, and the USA.

In the last years, European set theory has witnessed a renaissance with the creation of several new research groups and a substantial increase in the number of researchers. INFTY is an instrument that will consolidate already existing collaborations and establish new ones among the different European groups. The Programme aims to create a dynamic research and training network in order to ensure a leading position for European research on infinity and bring this subject to a higher level.

While there are several large European set theory research groups, there are also a number of small research groups scattered around Europe. Expanding existing collaborations into a global European network would integrate these groups and enable the creation of a European research space in the field. This will be achieved through conferences, workshops, summer schools for young researchers, and short-term scientific visits. These actions will make the European research environment more attractive for researchers on foundational and applied questions that involve infinity.

INFTY will promote the dialogue and collaboration between theoretical mathematicians, philosophers and specialists from applied sciences. Quick exchange of the latest results, new ideas, and interdisciplinary training and collaboration at a competitive level are absolutely necessary to maintain the leadership of European science in this area, which will yield European progress in fundamental and applicable research and the development of new methods and technologies.

The INFTY Programme extends and complements existing coordination and networking activities. The main aim of Europe-wide coordination is cross-fertilisation and exploration of the frontiers of the field. This is an intrinsically difficult task since it requires merging different traditions and scientific cultures. A way to achieve this aim is to facilitate a smooth convergence through a series of summer schools, workshops, and conferences. The workshops will be focused on specific frontiers with a potential to open new fields or solve specific problems.

Training from the more basic level is a requirement for a sustainable network. Set-theoretic methods are seldom taught in universities at the appropriate level to use them creatively. INFTY will provide financial support for courses or training activities and encourage transnational contact by sponsoring the participation of leading specialists from other countries as invited speakers.

INFTY participants will network with colleagues in other relevant research groups, to ensure that opportunities in the Programme's activities are known and open to other eligible participants. All proposed activities will be widely announced and will be open to all the relevant research community.

Participants in the ESF Research Conference The 2nd European Set Theory Meeting: In Honour of Ronald Jensen, Będlewo (Poland), 5-10 July 2009

Scientific Programme

The scientific context

Current mathematical research on infinity takes place mostly within set theory. It focuses on the following three major areas:

Large cardinals and inner model theory

While the ZFC formal system is sufficient for most of mathematics, it is incomplete. There are many different ways to extend ZFC, but one can compare these extensions by their consistency strength, measured by the scale of large cardinals. The goal of inner model theory is to provide canonical models for ever larger cardinals in the large-cardinal hierarchy. The main open problem is the construction of inner models for supercompact cardinals and beyond.

Descriptive set-theoretic methods in classification problems

Descriptive set theory studies definable infinite subsets of Polish spaces, such as the reals, the Cantor space, function spaces, etc. But recently, new important developments have linked it with the theory of dynamical systems, ergodic theory, and representation theory via the study of orbit equivalence relations and the corresponding quotient spaces, or the study of distal flows. The main object of this theory is to provide the framework and tools for comparing classification problems from various areas of mathematics.

Infinite combinatorics

This active field of mathematics deals mainly with combinatorial properties of infinite uncountable structures. Since many combinatorial questions are undecidable, the method of *forcing*, introduced by Cohen in the 1960s to prove the independence of the Continuum Hypothesis, plays a central role. Currently, the most active areas are the study of forcing axioms, statements which assert the saturation of the universe of sets under various forcing extensions, and transfinite cardinal arithmetic and Shelah's pcf theory, which searches for absolute properties.

Philosophical investigations involving infinity have gained a strong interest and momentum, sparked by recent dramatic advances on classical problems, such as Woodin's work on large cardinals and the Continuum Hypothesis, and Shelah's work on the Singular Cardinal Hypothesis.

Philosophical issues in the foundations of mathematics

Virtually all mathematical theorems today can be formulated as statements in the ZFC system. The truth of a mathematical statement is then nothing but the claim that the statement can be derived from the ZFC axioms using the rules of formal logic. This, however, does not explain several issues: why should we use these axioms and not some others, why should we employ the logical rules we do and not some others, why "true" mathematical statements (e.g., the laws of arithmetic) appear to be true in the physical world, and so on. Understanding the properties of infinite objects, as axiomatised in the ZFC system, is at the heart of these investigations and is therefore paramount to our understanding of the nature of mathematics and its effectiveness in the physical world. The applicability of set theory throughout mathematics is a consequence of its utmost generality. However, the possibility of possessing such a general framework is the result of deep conceptual analysis of philosopher-logicians in the early 20th century. Moreover, extending the reach of set theory to questions that until now have evaded solution can only succeed with the help of ever more penetrating philosophical analysis. An important part of the scientific programme of the INFTY network is to join the efforts of set theorists and philosophers in a fresh attempt to understand what is missing in the current axiomatisation of set theory.

Connections with other areas of mathematics and computer science

Powerful techniques of set theory are used successfully in other areas of mathematics sensitive to foundational issues. This is achieved by directly applying these tools to answer questions arising in abstract algebra, measure theory, general topology, ergodic theory, operator algebras, group representations, etc. In addition, these tools can be used to uncover hitherto unsuspected theories which are then studied by purely classical methods. An example is the numerous classical consequences of Shelah's pcf theory.

The INFTY network also includes a group of leading researchers in the area of modeling computation by means of sets. The basic framework and founding principles of such an approach to set theory are still in the making under the title of constructive set theory. By joining efforts within INFTY, the two approaches to set theory, constructive and non-constructive, can fully benefit from each other's scientific breakthroughs. It is already clear that the powerful set-theoretic methods of inner models, large cardinals and forcing have their counterparts in both approaches.

The universality of the language of set theory has been taken advantage of also by mathematical linguists, and although infinity plays little or no role in the main application areas of linguistics, the need to develop conceptually simple general theories of, for example, the semantics of natural language, has brought set theoretical methods to the limelight of general linguistics.

Scientific tasks

The scientific tasks of INFTY are the following:

A. The structure of definable subsets of the continuum

This task deals with the study of definable subsets of the real line (and general Polish spaces), its applications to various parts of mathematics (real and functional analysis, harmonic analysis, ergodic theory, dynamical systems, operator algebras), and its links with set theory (determinacy of games, large cardinals, and inner models). The main themes are:

- Hierarchies of definable sets: Wadge, Borel, and projective hierarchies;
- Determinacy of games and regularity properties of definable sets of real numbers;
- Complexity of "natural sets" in analysis;
- Classification of definable structures on the real numbers: linear and partial orderings, Polish groups actions, descriptive dynamics, and definable equivalence relations.

B. Infinite combinatorics

The tools of modern combinatorial set theory, such as combinatorial principles, partition calculus, infinite trees, ultrapowers, forcing axioms, and large cardinal axioms, have been very successful in resolving problems in areas such as general topology, abstract functional analysis, measure theory, and algebra. Some of the main topics we will pursue are:

- Forcing axioms. Applications of Martin's Axiom, the Proper Forcing Axiom, and Martin's Maximum to general topology, measure theory, and algebra. A recent success has been the solution of the L-space problem;
- Iterated forcing techniques. We are interested in the development of forcing techniques with the continuum bigger than $\aleph_2$;
- Infinitary languages and classification of uncountable structures;
- Cardinal arithmetic and large cardinals: The aim is to prove absolute upper bounds in cardinal arithmetic. A major open question is whether the set of possible cofinalities of a countable set of regular cardinals is necessarily countable. Shelah's theory of possible cofinalities (pcf) provides a new point of view and a powerful tool for applications in general topology, infinite permutation groups, Boolean algebras, etc.

C. Inner models of large cardinals and aspects of determinacy

The main work is on the Inner Model Programme, whose goal is to associate to each large cardinal axiom a minimal canonical model whose structure can be analysed in detail. These models provide evidence for the consistency of large cardinal axioms.

Our investigations concentrate in the following areas:

- Analysis of the fine structure of these models, and study of general iterability criteria, in particular a unifying approach to iterability past a Woodin cardinal;
- Construction of higher core models (i.e., admitting "many" Woodin cardinals);
- Combinatorial principles in inner models and their transfer to the surrounding universe;
- Aspects of determinacy: The goal is to explain the useful but ad hoc determinacy hypotheses and their numerous mathematically intriguing consequences by deriving them from strong axioms of infinity possessing intrinsic evidence.

($\Diamond_\kappa$) There is a sequence $\langle S_{\nu<\kappa}\rangle$ s.t. $S_\alpha \subseteq \alpha$ $(\alpha<\kappa)$ and for every $X \subset \kappa$, the set $\{\alpha \mid X\cap\alpha = S_\alpha\}$ is Mahlo in κ.

($\Diamond^+_\kappa$) There is a sequence $\langle S_{\nu<\kappa}\rangle$ s.t. $S_\alpha \subseteq \mathcal{P}(\alpha)$, $\overline{S_\alpha} \leq \overline{\alpha}$ $(\alpha<\kappa)$ and for every $X \subset \kappa$, the set $\{\alpha \mid X\cap\alpha \in S_\alpha\}$ is Mahlo in κ.

Ronald Jensen's diamond principles.

D. Applications of set theory to Banach spaces, algebra, topology and measure theory

Some examples of applications are the following:

- Infinite combinatorics, especially Ramsey combinatorics, in Banach space theory, and also in the study of partition regular systems of linear equations, generalising Van der Waerden's classical result on arithmetical progressions;
- Descriptive set-theoretic phenomena in topology, measure theory, and ergodic theory;
- Forcing constructions of Boolean algebras;
- Set-theoretic constructions of special subsets of the real line;
- Combinatorial methods in topology;
- Large cardinals in category theory and homological algebra;
- Characterisation of module-theoretic problems by cardinal properties. Construction of complicated modules over commutative rings using infinite combinatorics;
- Independence results in algebra, analysis and topology, e.g., consistency and independence of relatives of the Whitehead problem.

E. Constructive set theory and new models of computation

Some of the main topics are the following:

- The development of constructive/computational mathematics within constructive set theory, and also the metamathematics of constructive set theory;
- Sheaf and realisability models for Aczel's CZF, connections with type theory, computability, categorical logic (algebraic set theory), intuitionistic metamathematics;
- Computation on generalised machines with tapes or registers of infinite ordinal length;
- Formal mathematics; proof-checking systems with natural language interfaces, in collaboration with linguistics;
- Applications of set theory to the formal semantics of natural language, proof-checking systems with natural language interfaces, fuzziness in set theory and natural language;
- Well-founded rewriting systems, descriptive properties of languages accepted by Büchi automata and Linear Time Logic.

F. Set theory and philosophy

The general objectives are the applications of logical and mathematical methods in philosophy, as well as the analysis of philosophical questions arising in the exact study of infinity.

The search for new *axioms* beyond ZFC that are strong enough to resolve outstanding questions has been remarkably successful over the last 40 years producing, among other things, complete solutions to the classically unanswerable problems of second order number theory. With regard to third order number theory (which is where the Continuum Hypothesis belongs), a breakthrough was achieved only recently. What these results indicate is that the difficulties are not exclusively mathematical and any solution of the continuum problem will have to be accompanied by an analysis of what it means to be a solution. Here philosophical reflections on the nature of infinite mathematical objects and the concept of truth in mathematics are likely to play a role.

Our aim is to lay the groundwork for the above analysis and to map out possible directions it may take. Clearly this calls for a transdisciplinary approach. To this end, philosophers need to work together with set theorists and general mathematicians.

The following activities will be organised within the frame of INFTY:

- INFTY workshops (at least one focused workshop per year) and conferences (1st, 3rd and 5th year).
- Summer schools for young researchers and PhD students.
- Short visit grants (up to 2 weeks) and exchange grants (up to 3 months).

The first INFTY Steering Committee meeting was held in conjunction with the ESF Mathematics Conference in Partnership with EMS and ERCOM – The 2nd European Set Theory Meeting: In Honour of Ronald Jensen, on 5-10 July 2009 in Będlewo, Poland.

During the first year, INFTY is also giving support to the following activities:

- **ESI Workshop on Large Cardinals and Descriptive Set Theory.**
  14-27 June 2009. Vienna, Austria.
- **Mathematical Logic: Set theory and Model theory**.
  Autumn 2009. Institute Mittag-Leffler in Stockholm, Sweden.

Web page and publications

The INFTY web page http://www.inftynet.net contains a description of the Programme and information on all current activities, as well as instructions and deadlines for applications for short visit and exchange grants, Conferences, Workshops, and Summer Schools.

Proceedings of the major Conferences will be published.

Gregor Reisch, *Margarita Philosophica*, 1503-1508?. Typus Logice, presents the central methods and problems of logic.

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For the latest information on this Research Networking Programme consult the INFTY websites:
www.inftynet.net
and **www.esf.org/infty**

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