Logical Models for Reasoning about the Uncertainty of Many-Valued Events (A Very Gentle Overview)

Enrico Marchioni
(LoMoReVi)

Artificial Intelligence Research Institute (IIIA - CSIC)
Campus UAB, 08193 Bellaterra (Spain)
enrico@iiia.csic.es

joint work with T. Flaminio and L. Godo

15-18 September 2011
LogICCC Final Conference
Modelling Intelligent Interaction
Logic in the Humanities, Social and Computational Sciences

Berlin, Germany

The Main Goal

- To develop a uniform approach to define many-valued logics to represent uncertainty measures over many-valued events.

- Framework: (Δ-)core fuzzy logics (Cintula, Esteva, Godo, Gispert, Montagna, Noguera), i.e. expansions of the MTL logic (Esteva, Godo).

- We will (very) loosely refer to them as “many-valued logics”.

Measures of Uncertainty I

- Aim at formalizing the strength of our beliefs in the occurrence of some events.
- Assign to events a degree of belief concerning their occurrence.
- Different uncertainty models: probability, possibility, imprecise probabilities, belief functions, etc.
- In general, given a Boolean algebra **B** an uncertainty measure (capacity, fuzzy measure, plausibility measure) is a mapping $\mu : B \rightarrow [0, 1]$ such that $\mu(\bot) = 0$, $\mu(\top) = 1$, and $x \leq y$ implies $\mu(x) \leq \mu(y)$.

- (Finitely Additive) Probability:

$$\text{for all } x, y \in B,\ x \wedge y = \bot \text{ then } \mu(x \vee y) = \mu(x) + \mu(y).$$

- Possibility Measures (Zadeh 1978):

$$\mu(x \vee y) = \max(\mu(x), \mu(y))$$

- Necessity Measures (Zadeh 1978):

$$\mu(x \wedge y) = \min(\mu(x), \mu(y))$$

- Belief Functions (Shafer 1975):

$$\mu\left(\bigvee_{i=1}^{n} x_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mu\left(\bigwedge_{i \in I} x_i\right).$$

Logics of Uncertainty: The Classical Treatment

- Expansions of Classical Logic with Modal Operators.
- [Several Treatments]$P_{\leq\alpha}\phi$ ($P_{\geq\alpha}\phi$, $P_{=\alpha}\phi$): The probability of ϕ is less than (greater than, equal to) α.
- [Halpern] $P\phi$: Classical Logic is expanded with a axioms for (linear) polynomial inequalities. Probabilty is axiomatized in this context:

$$P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) = P(\varphi)$$

- Two-valued (modal) logics, Kripke models with a finitely additive probability measure.
- [Halpern] Treatment of possibility measures and belief functions.

Uncertainty of Many-Valued Events

- What about the uncertainty of many-valued events (statements)?
- We mean statements whose truth comes in degrees.
- Statements are many-valued, so they generally do not satisfy the laws of Boolean algebras.
- Algebras of many-valued logics: MV-algebras, Gödel algebras, etc. (generalizations of Boolean algebras)
- In general, given an algebra **A** of some many-valued logic, a *generalized plausibility measure* is a mapping $\mu : A \to [0, 1]$ such that $\mu(\bot) = 0$, $\mu(\top) = 1$, and $x \leq y$ implies $\mu(x) \leq \mu(y)$.

Uncertainty of Many-Valued Events: Measure-Theoretic Characterizations

- Probability over MV, Gödel algebras: Mundici, Kroupa, Panti, Aguzzoli, Gerla, Marra
- Possibility over MV, Gödel algebras: Dellunde, Flaminio, Godo, M.
- Belief functions over MV-algebras: Kroupa, Flaminio, Godo, M.
- Upper probabilities over MV-algebras: Fedel, Keimel, Montagna, Roth.

Many-Valued Logics of Uncertainty

- First suggested by Hájek and Harmancová (1994), and later later followed by Hájek, Esteva and Godo (1995).
- $\Psi :=$ "The proposition φ is plausible (probable, believable)".
- The degree of truth of Ψ can be interpreted as the degree of uncertainty of the proposition φ.
- The higher our degree of confidence in φ, the higher the degree of truth of Ψ.
- The predicate "is plausible (believable, probable)" can be regarded as a *many-valued* modal operator over the proposition φ.

Compatibility

- Several many-valued logics ((Δ-)core fuzzy logics) have a real-valued semantics.
- The interpretation of connectives corresponds to real-valued functions.
- A many-valued logic $\mathcal{L}$ is *compatible* with a class of uncertainty measures $\mathscr{M}$, whenever the operations (and relations) used to define the properties of $\mathscr{M}$ are definable in $\mathcal{L}$
- For instance, expansions of Łukasiewicz logic allow the expression of (bounded) sum $x \oplus y = \min(x + y, 1)$ and (truncated) subtraction $x \ominus y = \max(x - y, 0)$
- So, Łukasiewicz logic is compatible with the class of finitely additive probabilities.

Logics: Formal Definition

- Take a class of uncertainty measures $\mathscr{M}$
- Take the many-valued logic $\mathcal{L}_1$ for events
- Take some many-valued logic $\mathcal{L}_2$, compatible with $\mathscr{M}$
- Non-modal formulas ϕ, ψ are formulas of $\mathcal{L}_1$
- Atomic modal formulas are formulas of the form $\mathscr{M}\phi$, with $\phi \in \mathcal{L}_1$
- Modal formulas are built from atomic modal formulas using the connectives of $\mathcal{L}_2$
- Semantics is given by Kripke models equipped with a measure $\mu \in \mathscr{M}$

Theorem

Let $\mathcal{L}_1$ be a logic for events, and let $\mathcal{L}_2$ be a logic compatible with generalized plausibility measures. Then the following hold:

(1) *If $\mathcal{L}_1$ is locally finite*, and $\mathcal{L}_2$ enjoys FSRC, then $\mathrm{F}\mathscr{P}\mathscr{L}(\mathcal{L}_1, \mathcal{L}_2)$ is real-FSC.*

(2) *If $\mathcal{L}_2$ has SRC, then $\mathrm{F}\mathscr{P}\mathscr{L}(\mathcal{L}_1, \mathcal{L}_2)$ is real-SC.*

(3) *If $\mathcal{L}_2$ has FSRC, then $\mathrm{F}\mathscr{P}\mathscr{L}(\mathcal{L}_1, \mathcal{L}_2)$ is hyperreal-SC.*

*A many-valued logic $\mathcal{L}$ is said to be *locally finite* iff for every finite set V_0 of propositional variables, the Lindenbaum-Tarski algebra Fm_{V_0} of $\mathcal{L}$ generated by the variables in V_0 is a finite algebra.

Uncertainty of Classical Events

- Probability Measures: Esteva, Godo, Hájek
- Possibility and Necessity Measures: Hájek
- Belief Functions: Esteva, Godo, Hájek
- Lower and Upper Probabilities: M.
- Non-standard Probability Measures: Flaminio, Montagna
- Measures of Conditional Events: Godo, M.
- Uniform Treatment of Uncertainty Measures over Boolean Events: M.

- Probability Measures of Łukasiewicz Events: Flaminio, Godo
- Possibility and Necessity Measures of Łukasiewicz Events: Flaminio, Godo, M.
- Possibility and Necessity Measures of Gödel Events: Dellunde, Godo, M.
- Belief Functions of Łukasiewicz Events: Flaminio, Godo, M.
- Upper Probabilities of Łukasiewicz Events: Fedel, Hosni, Montagna

Coherence I

- Take a finite set of events $\phi_1, \ldots, \phi_k \in \mathcal{L}_1$, and a map $\mathbf{a} : \phi_i \mapsto \alpha_i \in [0, 1]$.

  Can the map **a** be extended to a generalized plausibility measure on the algebra generated by the formulas $\phi_1, \ldots, \phi_k$?

- Generalization of the classical de Finetti Coherence problem.

Definition

Let $\phi_1, \ldots, \phi_k$ be formulas in the language of $\mathcal{L}_1$ and let $\mathscr{M}$ be a class of generalized plausibility measures. Then a map $\mathbf{a} : \{\phi_1, \ldots, \phi_k\} \to [0, 1]$ is said to be:

(*i*) A *rational* assignment, provided that for every $i = 1, \ldots, k$, $\mathbf{a}(\phi_i)$ is a rational number.

(*ii*) *$\mathscr{M}$-Coherent* if there is an uncertainty measure $\mu \in \mathscr{M}$ on the Lindenbaum-Tarski algebra Fm_V generated by the variables occurring in $\phi_1, \ldots, \phi_k$, such that, for all $i = 1, \ldots, n$, $\mathbf{a}(\phi_i) = \mu([\phi_i])$.

Coherence II

- We take $\mathrm{R}Ł$ as $\mathcal{L}_2$. $\mathrm{R}Ł$, Rational Łukasiewicz logic, is an expansion of Łukasiewicz logic that allows to define rationals in the language.

Theorem

Let $\phi_1, \ldots, \phi_k$ be formulas in $\mathcal{L}$, and let

$$\mathbf{a} : \phi_i \mapsto \frac{n_i}{m_i}$$

be a rational assignment. Then the following are equivalent:

(i) $\mathbf{a}$ *is* $\mathscr{M}$*-coherent,*

(ii) *The modal theory* $\Gamma = \{M(\phi_i) \leftrightarrow \overline{n_i/m_i} \mid i = 1, \ldots, k\}$ *is consistent in* $\mathrm{F}\mathscr{M}(\mathcal{L}_1, \mathrm{R}Ł)$ *(i.e.* $\Gamma \nvdash_{\mathrm{F}\mathscr{M}} \perp$*).*

Flaminio T., Godo L., Marchioni E. - Reasoning about uncertainty of fuzzy events: an overview. In *Reasoning under Vagueness - Logical, Philosophical, and Linguistic Perspectives*, Cintula P., Fermüller C., Godo L., Hájek P. (Editors), Studies in Logic, College Publications, forthcoming.

THANKS!