Complexity Results for Dependence Logic

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Outline of the talk

We discuss recent complexity results on dependence logic and its variants. We focus on:

- The expressive power and complexity of certain natural syntactic fragments of dependence logic.
- The complexity of certain extensions of dependence logic.

Motivations for this line of research include:

- understanding the computational content of logical operators and constructs can help us with finding the most appropriate logical tools for modeling.
- for applicability, it is very useful to know, just by looking at a formula it's approximate complexity.

Dependence logic

Definition

The syntax of $\mathcal{D}$ extends the syntax of FO by new atomic dependence formulas

$$=\!(t_1, \ldots, t_n), \tag{1}$$

where $t_1, \ldots, t_n$ are terms.

In (1), n is called the *width* of the dependence atom.

Basic properties of $\mathcal{D}$

For sentences:

Theorem

$\mathcal{D} = \mathrm{ESO} = \mathrm{NP}$.

The second equality holds over finite structures.

We will next look at the expressive power and complexity of certain syntactic fragments of $\mathcal{D}$. Such fragments can be defined by:

1. restricting the number of variables or quantifiers in formulas,
2. the width n of the dependence atoms $=(x_1, \ldots, x_n, y)$,
3. restricting the quantifier nestings in formulas.

Complexity of fragments of $\mathcal{D}$

The following result indicates that new methods and ideas might be needed in order to understand the complexity of such fragments:

Theorem (Jarmo Kontinen; 2010)

Define

1. $\varphi \equiv =\!(x, y) \vee =\!(z, u)$
2. $\psi \equiv =\!(x, y) \vee =\!(z, u) \vee =\!(z, u)$

Deciding whether $\mathfrak{A}$ and X satisfies φ is NL-complete and, for ψ, NP-complete.

Restricting the number of variables

Results on the 2-variable fragment of $\mathcal{D}$

Denote by $\mathcal{D}^2$ the sentences of $\mathcal{D}$ using only two variables.

Theorem (K., Kuusisto, Lohmann, and Virtema; 2011)

1. *The Satisfiability (and Finite Satisfiability) problem of $\mathcal{D}^2$ is NEXPTIME-complete.*
2. *The logic $\mathcal{D}^2$ is quite expressive being able to express, e.g., "$\mathfrak{A}$ infinite" and "$|P| = |Q|$".*
3. *In contrast, the satisfiability (and finite satisfiability) problem of IF^2 is undecidable.*

Remark

Jonni Virtema (Univ. Tampere) recently observed that $\mathcal{D}^2$ can express NP-complete problems.

Restricting the number of universal quantifiers/the width of dependence atoms

Definition

Let $k \in \mathbb{N}^*$.

- $\mathcal{D}(k - \forall)$ consists of those sentences ϕ of $\mathcal{D}$ having at most k occurrences of $\forall$ (no reusing of variables).
- $\mathcal{D}(k - dep)$ consists of sentences ϕ of $\mathcal{D}$ in which dependence atoms of width at most $k + 1$ appear.

The case of $\mathcal{D}(k - dep)$

Definition

Denote by $\mathrm{ESO}_f(k\text{-ary})$ the class of ESO-sentences

$$\exists f_1 \ldots \exists f_n \psi,$$

in which the function symbols f_i are at most k-ary and ψ is a FO-formula.

Theorem (Durand and K.; 2011)

Let $k \in \mathbb{N}^*$. $\mathcal{D}(k - dep) = \mathrm{ESO}_f(k\text{-ary})$.

The case of $\mathcal{D}(k - \forall)$

Definition

Denote by $\mathrm{ESO}_f(k\forall)$ the class of ESO-sentences in Skolem Normal Form

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_r \psi,$$

where $r \leq k$ and ψ is quantifier-free.

Theorem (Durand and K.; 2011)

Let $k \in \mathbb{N}^*$. Then

$$\begin{aligned} \mathrm{NTIME}_{\mathrm{RAM}}(n^k) = \mathrm{ESO}_f(k\forall) \leq \mathcal{D}(2k - \forall) \quad &\leq \quad \mathrm{ESO}_f(2k\forall) \\ &= \quad \mathrm{NTIME}_{\mathrm{RAM}}(n^{2k}). \end{aligned}$$

The equality $\mathrm{NTIME}_{\mathrm{RAM}}(n^k) = \mathrm{ESO}_f(k\forall)$ is due to Grandjean and Olive (2004).

Hierarchy theorems

Theorem (Durand and K.; 2011)

If τ has a $k+1$-ary R, then $\mathcal{D}(k-dep)[\tau] \subsetneq \mathcal{D}(k+1-dep)[\tau]$.

Theorem (Durand and K.; 2011)

For $k \geq 1$ and any vocabulary:

1. $\mathcal{D}(k-\forall) \subseteq \mathcal{D}(k-dep)$,
2. $\mathcal{D}(k-\forall) \subsetneq \mathcal{D}(k+1-dep)$,
3. $\mathcal{D}(k-\forall) \subsetneq \mathcal{D}(2k+2-\forall)$.

Extensions of dependence logic

We briefly discuss three different types of extensions of dependence logic.

Theorem (Abramsky and Väänänen; 2009, Yang; 2010)

The extension of dependence logic by the intuitionistic implication is equivalent to full $\mathrm{SO} = \mathrm{PH}$.

Above, PH is the complexity class the *Polynomial Hierarchy*.

Theorem (Grädel and Väänänen; 2010)

Independence logic is equivalent to $\mathrm{ESO} = \mathrm{NP}$ *for sentences.*

Extensions of dependence logic cont.

Let $\mathcal{D}(\mathrm{M})$ be the extension of $\mathcal{D}$ by the following majority quantifier:

$$\mathfrak{A} \models_X \mathrm{M}x\phi(x) \text{ iff for at least } |A|^{|X|}/2 \text{ many function } F\colon X \to A \text{ we have } \mathfrak{A} \models_{X(F/x)} \phi(x).$$

Theorem (Durand, Ebbing, K., and Vollmer; 2011)

$\mathcal{D}(\mathrm{M}) = \mathrm{CH}$.

Above, CH is the complexity class the *Counting Hierarchy* that contains PH and full second-order logic.