On Mathematical Fuzzy Logic

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A little bit of history: fuzzy logic

- Zadeh 1965
- Goguen 1967
- Lee 1972
- Mamdani 1974
- Bandler, Kohout 1980
- Trillas, Valverde 1985
- Klir, Folger 1988
- Zimmermann 1991

⋮

A little bit of history: many-valued logic

- Łukasiewicz 1920
- Łukasiewicz - Tarski 1930
- Gödel 1932
- Moisil 1940
- Rose - Rosser 1958
- Chang 1959
- Dummett 1959
- Belluce - Chang 1963
- Horn 1969
- Ragaz 1981
- Mundici 1987, 1993
- Gottwald 1988

⋮

A little bit of history: fuzzy logic

- Pavelka 1979
- Pulr 1984
- Takeuti-Titani 1984, 1992
- Novák 1990
- Gottwald 1993
- Hájek-Esteva-Godo 1996

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WOS: **1150+** Scopus: **1150+**

MathSciNet **200+** Google Scholar: **2200+**

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Subject classification 03

Publication Type All

Citations to display 10

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Classification expansion for '03B52'

Primary Classification

03 (1980-now) Mathematical logic and foundations
03B (1980-now) General logic
03B52 (1980-now) Fuzzy logic; logic of vagueness [See also 68T27, 68T37, 94D05]

Citations	Publication
570	**MR1321597 (96e:03057)** Kechris, Alexander S. Classical descriptive set theory. Graduate Texts in Mathematics, 156. *Springer-Verlag, New York,* 1995. xviii+402 pp. ISBN: 0-387-94374-9 (Reviewer: Jakub Jasiński), 03E15 (03-01 03-02 04A15 28A05 54H05 90D44)
442	**MR0882921 (88m:03003)** Soare, Robert I. Recursively enumerable sets and degrees. A study of computable functions and computably generated sets. Perspectives in Mathematical Logic. *Springer-Verlag, Berlin,* 1987. xviii+437 pp. ISBN: 3-540-15299-7 (Reviewer: Peter G. Hinman), 03-02 (03D20 03D25 03D30)
372	**MR0597342 (82f:03001)** Kunen, Kenneth Set theory. An introduction to independence proofs. Studies in Logic and the Foundations of Mathematics, 102. *North-Holland Publishing Co., Amsterdam-New York,* 1980. xvi+313 pp. ISBN: 0-444-85401-0 (Reviewer: U. Felgner), 03-02 (03E35 03Exx 04-02)
324	**MR1221741 (94e:03002)** Hodges, Wilfrid Model theory. Encyclopedia of Mathematics and its Applications, 42. *Cambridge University Press, Cambridge,* 1993. xiv+772 pp. ISBN: 0-521-30442-3 (Reviewer: J. M. Plotkin), 03-01 (03-02 03Cxx)
323	**MR0506523 (80a:03062)** Jech, Thomas Set theory. Pure and Applied Mathematics. *Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London,* 1978. xi+621 pp. ISBN: 0-12-381950-4 (Reviewer: E. Mendelson), 03Exx (04-02)
228	**MR1633348 (99j:03001)** van den Dries, Lou Tame topology and o-minimal structures. London Mathematical Society Lecture Note Series, 248. *Cambridge University Press, Cambridge,* 1998. x+180 pp. ISBN: 0-521-59838-9 (Reviewer: O. V. Belegradek), 03-02 (03C50 03C60 14P10 52-02 54-02 55-02 57-02)
208	**MR1900263 (2003c:03048)** Hájek, Petr Metamathematics of fuzzy logic. Trends in Logic—Studia Logica Library, 4. *Kluwer Academic Publishers, Dordrecht,* 1998. viii+297 pp. ISBN: 0-7923-5238-6 (Reviewer: Siegfried Johannes Gottwald (Zbl 0937.03030)), 03B52 (03-02 03B50 03D35 03G10)
208	**MR1059055 (91c:03026)** Chang, C. C.; Keisler, H. J. Model theory. Third edition. Studies in Logic and the Foundations of Mathematics, 73. *North-Holland Publishing Co., Amsterdam,* 1990. xvi+650 pp. ISBN: 0-444-88054-2, 03Cxx (03-01 03C57 03C95 03H05)
193	**MR1350295 (96k:03002)** Bartoszyński, Tomek; Judah, Haim Set theory. On the structure of the real line. *A K Peters, Ltd., Wellesley, MA,* 1995. xii+546 pp. ISBN: 1-56881-044-X (Reviewer: Eva Coplakova), 03-02 (03Exx)
182	**MR0899269 (89m:03057)** Girard, Jean-Yves Linear logic. *Theoret. Comput. Sci.* 50 (1987), no. 1, 101 pp. (Reviewer: G. E. Mints), 03F50 (03B70 68Q99)

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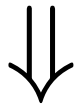

Mathematical fuzzy logic

Łukasiewicz logic Ł

Connectives: implication $\to$ and 'falsum' $\bot$ (we set $\neg\varphi = \varphi \to \bot$)

(Standard) semantics: evaluation is a mapping $e\colon \mathrm{FOR} \to [0,1]$ st:

$$e(\bot) = 0 \qquad\qquad e(\varphi \to \psi) = \min\{1, 1 - e(\varphi) + e(\psi)\}$$

Tautologies: formulae *always* evaluated to 1

Theorems: formulae derived (in the usual way) using *Modus Ponens* (from φ and $\varphi \to \psi$ infer ψ) from the axioms:

$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$$

Completeness: $\mathrm{Thm}(Ł) = \mathrm{Taut}(Ł)$

Multitude of fuzzy logics

Fuzzy logics in 'full language' $(\rightarrow, \rightsquigarrow, \&, \wedge, \vee, 0, 1, \bot, \top)$: Ł, G, Π, BL, SBL, WCBL, MTL, IMTL, SMTL, WNM, NM, WCMTL, ΠMTL, psMTL^r, psBL^r, UL, IUL, ULM, psUL, ...

We have families of their extensions: L_n, $\mathsf{L}_{\leq n}$, $\mathsf{C}_n\mathsf{L}$, $\mathsf{S}_n\mathsf{L}$, ...

We can add truth constants: just distinguished ones like $\frac{1}{2}$, all rationals, or from a set $\mathcal{C} \subseteq [0, 1]$

We can add unary connectives: Δ, $\sim$, δ_n, vt, st, ...

We can add binary connectives: $\odot$, $\rightarrow_\Pi$, ...

We can study fragments: pure implicational, $\bot$-free, ...

Consequence relations

Semantically: $T \models_{\text{Ł}} \varphi$ if for each evaluation s.t. $e[T] \subseteq \{1\}$ also $e(\varphi) = 1$

Syntactically: $T \vdash_{\text{Ł}} \varphi$ if there is a proof using formulae of T as special axioms

1st problem: equivalence '$T \models_{\text{Ł}} \varphi$ iff $T \vdash_{\text{Ł}} \varphi$' holds for *finite* T only

2nd problem: consider rule from $\varphi \to \neg\varphi$ and $\neg\varphi \to \varphi$ infer $\bot$.

- $\varphi \to \neg\varphi, \neg\varphi \to \varphi \not\vdash_{\text{Ł}} \bot$
- if we add it as a new inference rule to Łukasiewicz logic we get no new theorems

PC, G. Metcalfe. *Structural Completeness in Fuzzy Logics.* Notre Dame J. of Formal Logic, 50(2):153–183, 2009

Abstract study of fuzzy logics: semilinearity

PC, C. Noguera. *Implicational (semilinear) logics I: a new hierarchy.* Archive for Mathematical Logic, 49:417–446, 2010

Fact: any logic (as consequence relation) has its algebraic (matrix) semantics

An implicational logic has 'nice' implication connective which defines order in its algebras

A semilinear logic is complete w.r.t. its algebras which are linearly ordered

It a mathematical *approximation* of the notion of fuzzy logic as used in the literature.

Abstract study of fuzzy logics: density

Being complete w.r.t. all chains need not be enough: we sometimes want completeness w.r.t. better classes of chains (in extreme w.r.t. just one particular algebra as in the case of Łukasiewicz logic)

A. Ciabattoni, G. Metcalfe. *Density elimination.* Theoretical Computer Science, 403(2-3):328–346, 2008

Adding truth constants

F. Esteva, L. Godo, C. Noguera. *First-order t-norm based fuzzy logics with truth-constants: distinguished semantics and completeness.* Annals of Pure and Applied Logic, 161:185–202, 2009

Z. Haniková. *Logics with truth constants for delimiting idempotents.* Submitted

First-order fuzzy logic

Decidability and arithmetical hierarchy

- F. Montagna, C. Noguera. *Arithmetical complexity of first-order predicate fuzzy logics over distinguished semantics.* Journal of Logic and Computation, 20:399–424, 2010
- P. Hájek. *Comments on interpretability, decidability and other topics on fuzzy logic.* To appear in J. of Logic and Comp.
- M. Baaz, A. Ciabattoni, N. Preining. *SAT in Monadic Goedel Logics: a borderline between decidability and undecidability.* In Proc. of the 16th LLIC, LNCS, 5514:113–124, 2009.

Model theory

- P. Dellunde. *Preserving Mappings in Fuzzy Predicate Logics.* To appear in Journal of Logic and Computation.
- M. Baaz, G. Metcalfe. *Herbrand Theorems and Skolemization for Fuzzy Logics.* In Proceedings of CiE 2008, LNCS vol, 5028, pp. 22–31, Springer, 2008

First-order fuzzy logic cont.

Proof theory

- M. Baaz, A. Ciabattoni, C. Fermüller. *Cut Elimination for First Order Gödel Logic by Hyperclause Resolution* In Proceedings of the 15th LPAR, LNCS, vol. 5530, pp. 451–466, 2008
- A. Ciabattoni, N. Galatos, K. Terui. *From Axioms to Analytic Rules in Nonclassical Logics.* In Logic in Computer Science, pp. 229–240, IEEE, 2008

Fuzzy description logics

- F. Bobillo, F. Bou, U. Straccia. *On the Failure of Finite Model Property in some Fuzzy Description Logics.* Fuzzy Sets and Systems, 171:1–12, 2011
- A. Garcia-Cerdańa, E. Armengol, F. Esteva. *Fuzzy Description Logics and t-norm based Fuzzy Logics.* To appear in Int. J. of Approximate Reasoning.

Modalities in fuzzy logics

- P. Hájek. *On the fuzzy modal logic S5(C).* To appear in Fuzzy Sets and Systems.
- F. Bou, F. Esteva, L. Godo, R. Rodriguez. *On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice.* To appear in Journal of Logic and Computation, 2010
- G. Metcalfe, N. Olivetti. *Proof Systems for a Gödel Modal Logic.* In proc. of TABLEAUX 2009, LNCS, vol. 5607, pp. 265-279, 2009

Handbook of Mathematical Fuzzy Logic

- edited by: P. Cintula, P. Hájek, C. Noguera
- to be publish in College Publication
- 11 chapters by 19 authors (11 from LoMoReVI!)
- 900+ pages in 2 volumes
- purely mathematical and theoretical work: no philosophy, linguistics, or applications.

CONTENTS