Modelling Resource Allocation in Linear Logic

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Overview

- Resource allocation problems: combinatorial auctions;
- Logical modelling of preferences;
- A proof-theoretic approach: Linear logic, a constructive account of proofs;
- Combinatorial auctions on multi-sets of goods;
- Reasoning, structural rules, and allocation problems.

This presentation is based on Porello and Endriss *KR 2010* and *ECAI 2010*.

Combinatorial auctions

- Given a *set* of goods $\mathcal{G}$ and a set of bidders $\mathcal{N}$;
- Bidders evaluate bundles of goods $S \subseteq \mathcal{G}$ offering *atomic bids* of the form (S, w) where w is the price associated to the bundle S.
- Atomic bids (S, w) define utility functions $v_S : \mathcal{P}(\mathcal{G}) \to W$, where W is a set of values:

$$S' \subseteq \mathcal{G},\ v_S(S') = w \text{ if } S \subseteq S',\ v_S(S') = 0 \text{ otherwise.}$$

- Languages for complex bids (Nisan, 2006).
- The *value* of an allocation α is given by $v(\alpha) = \sum_i \{w_i \ : \ (S_i, w_i) \in \alpha\}$
- *Winner determination problem* (WDP): finding an allocation that maximizes the revenue given a set of bids.
(Usually NP-Complete, reduction form SET PACKING)

Types of goods

- The matching between demand and offer can be modelled with logic: by viewing goods as propositional atoms, and preferences as logical formulas we have:

$$v_{a \wedge b}(\{a, b, c\}) = w \text{ iff } \{a, b, c\} \models a \wedge b$$

(Weighted formula, Goal bases).

- If goods are available in multi-sets, or lists, classical entailment is problematic:
- Structural rules in sequent calculus:

Weakening	*Contraction*	*Exchange*
$\dfrac{\{a, b\} \vdash a \wedge b}{\{a, a, b\} \vdash a \wedge b}$	$\dfrac{\{a, a\} \vdash a \wedge a}{a \vdash a \wedge a}$	$\dfrac{\{a, b\} \vdash a \wedge b}{\{b, a\} \vdash a \wedge b}$

- Which notion of logical consequence is suitable in such cases?
- Linear logic provides a good canddate since it is capable of controlling the application of structural rules.

Combinatorial auctions on multi-sets of goods

- A finite multiset of goods $\mathcal{M}$ (with finite multiplicity).
- The *atoms* $\mathcal{A} = \{p_1, \ldots, p_m\}$ are the elements of $\mathcal{M}$. Multisets of goods can be defined using the *tensor* conjunction $\otimes$ in *Linear Logic*. E.g. $p \otimes p \otimes q$.

Atomic bids are implications $B \multimap u^k$ ("if you give me B, I give u^k"):
B is a tensor product of atoms in $\mathcal{A}$,
u^k is used to model prices symbolically as tensors of a given unit symbol u:
$u^k = u \otimes \underbrace{\ldots}_{k-times} \otimes u$

$$\underbrace{p, q, r}_{goods}, \underbrace{p \otimes q \otimes r \multimap u^k}_{bid} \vdash u^k$$

Weakening can be used to model (global) *Free Disposal Assumption* (a bidder is willing to obtain *at least* what she demands).

Valuations as formulas. Allocations as proofs

- We can define classes of bidding languages using fragments of linear logic, including the usual language (OR, XOR, Goal Bases).
- Moreover we can express much more: e.g. the distinction between *sharable* and *non-sharable* (or *reusable*) resources: $!(a \otimes b) \otimes c$.
(! local structural rules).
- *Valuations as formulas*: formulas BID generate utility functions v_{BID} mapping multi-sets $X \subseteq \mathcal{M}$ to values:

$$v_{\mathrm{BID}}(X) = \max\{k \mid X, \mathrm{BID} \vdash u^k\}$$

- *Allocations as proofs*: we can use proof search to deal with allocation problems

Theorem [Porello and Endriss, *KR 2010*]

A proof in (fragments of) linear logic corresponds to an allocation of goods and *vice versa*.

Example

WDP (decision version):
Given goods: p, q, q, bids: $p \otimes q \multimap u^4$ and $q \multimap u^2$, can we get a revenue of 6 units (u^6)? Can we prove the following sequent?

$$\underbrace{p, q, q}_{goods}, \underbrace{p \otimes q \multimap u^4}_{bid1}, \underbrace{q \multimap u^2}_{bid2} \vdash u^6$$

$$\dfrac{\dfrac{p, q \vdash p \otimes q \qquad u^4 \vdash u^4}{p, q, \underbrace{p \otimes q \multimap u^4}_{bid1} \vdash u^4} \multimap L \qquad \dfrac{q \vdash q \qquad v \vdash v}{q, \underbrace{q \multimap u^2}_{bid2} \vdash u^2} \multimap L}{p, q, q, p \otimes q \multimap u^4, q \multimap u^2 \vdash u^4 \otimes u^2 (= u^6)} \otimes R$$

- The proof shows that a given value is achievable.

Structural rules, reasoning methods and allocation problems

We can picture the following correspondence:

Structural Rules	*Logic*	*Allocation problem*
W, C, E	Classical Logic	Sets, quantities of types of good do not matter
W, E	Affine Logic	Multi-sets, with Free Disposal
E	LL	Multi-sets, without Free Disposal
-	NCLL (Lambek calculus)	Lists of goods

- Negotiation problem can be approached in a similar way. (Porello and Endriss *ECAI 2010*)

Conclusion

- We presented a model of resources allocation based on the constructive treatment of proofs.
- Linear logic allows for expressing valuations (utility functions) as formulas and to view allocations as proofs.
- In the treatment we proposed, we used the Horn fragment of LL for which proof-search complexity is NP complete.
- The bundle of goods the auctioneer owns is represented (basically) by the suitable *conjunction* of goods.
  (E.g. $\{p, q, q\} \equiv p \otimes q \otimes q$).
- Future work in this direction include the case in which the bundle the auctioneer owns is given by a general formula that represent the relations between the goods the auctioneer puts on the bundle to sell.