# Social Networks: Influence and Centrality

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  - some concepts of influence in social networks.

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$$g_{ij} = \begin{cases} 1 & \text{if there is a link between } i \text{ and } j \\ 0 & \text{otherwise,} \end{cases}$$

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- $d_i(g)$ = degree of $i$ in $g$ = number of $i$'s neighbors in $g$, i.e.,

$$d_i(g) = |N_i(g)|$$

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  If there is no path between $i$ and $j$ in $g$, we set $d(i, j; g) = \infty$.
- $g^k = k$th power of $g$; $g^0 := \mathbb{I}$ with $\mathbb{I} = n \times n$ identity matrix, where
  $g^k_{ij} =$ number of walks of length $k$ that exist between $i$ and $j$ in $g$.

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- For extended surveys, see e.g. Jackson (2008), Goyal (2007), Wasserman & Faust (1994), Freeman (1979), Everett & Borgatti (2005).

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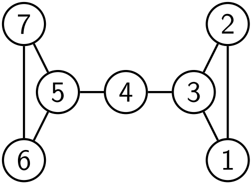


$C_i^d(g) = 0.5$ for $i \in \{3,5\}$, $C_i^d(g) = 0.33$ for $i \notin \{3,5\}$.

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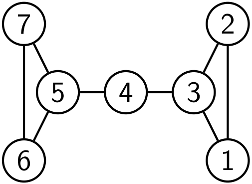


$C_4^c(g) = 0.60$, $C_3^c(g) = C_5^c(g) = 0.55$, $C_i^d(g) = 0.4$ otherwise.

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- The betweenness centrality $C_i^b(g)$ of node $i$ in network $g$ is

$$C_i^b(g) = \frac{2}{(n-1)(n-2)} \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(kj)}{P(kj)}$$

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  $P(kj)$ = total number of geodesics between $k$ and $j$
- Index of the potential of a node for control of communication: the possibility to intermediate in the communications of others is of importance.

# Betweenness centrality of a node (2/2)

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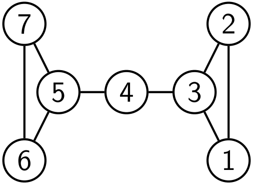


$C_4^b(g) = 0.60$
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$C_i^b(g) = 0$ for $i \in \{1, 2, 6, 7\}$

# Katz prestige

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- Calculating $C^{PK}(g)$ - finding the unit eigenvector of $\widetilde{g}$:

$$C^{PK}(g) = \widetilde{g}\, C^{PK}(g)$$

$$(\mathbb{I} - \widetilde{g})C^{PK}(g) = \mathbf{0}$$

$\widetilde{g}$ - the normalized adjacency matrix $g$ with $\widetilde{g}_{ij} = \frac{g_{ij}}{d_j(g)}$,
we set $\widetilde{g}_{ij} = 0$ for $d_j(g) = 0$.
$C^{PK}(g)$ - the $n \times 1$ vector of $C_i^{PK}(g)$, $i \in N$.

# Second prestige measure of Katz

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- Introducing an attenuation parameter $a$ to adjust the measure for the lower 'effectiveness' of longer walks in a network.
- The prestige of a node is a weighted sum of the walks that emanate from it, and a walk of length $k$ is of worth $a^k$, where $0 < a < 1$. The vector of prestige of nodes is

$$C^{PK2}(g, a) = ag\mathbf{1} + a^2 g^2 \mathbf{1} + \cdots + a^k g^k \mathbf{1} + \cdots$$

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- For $a$ sufficiently small, $C^{PK2}(g, a)$ is finite and

$$C^{PK2}(g, a) - agC^{PK2}(g, a) = ag\mathbf{1}$$

$$C^{PK2}(g, a) = (\mathbb{I} - ag)^{-1} ag\mathbf{1}.$$

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$$C^B(g, a, b) = ag\mathbf{1} + abg^2\mathbf{1} + \cdots + ab^k g^{k+1}\mathbf{1} + \cdots$$

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  where $a$ and $b$ are parameters, and $b$ is sufficiently small.
  - $b$ captures how the value of being connected to another node decays with distance.
  - $a$ captures the base value on each node.

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- A two-parameter family of prestige measures which can be seen as a direct extension of $C^{PK2}(g, a)$.
- An agent can have some status which does not depend on its connections to others.
- Bonacich centrality (Bonacich (1987)) is given by

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  where $a$ and $b$ are parameters, and $b$ is sufficiently small.
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  - Obviously $C^{PK2}(g, a)$ and $C^B(g, a, b)$ coincide when $a = b$.

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# The yes-no (one-step) model of influence

SOCIAL NETWORK, PLAYERS, INFLUENCE

| | | | | |
|---|---|---|---|---|
| inclinations $i$ ('yes' or 'no') | $\to \to$ | decisions $Bi$ influence function $B$ | $\to$ | group decision $gd(Bi)$ |

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision
- An agent has an inclination to say either YES (+1) or NO (-1)
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$
- $I = \{+1, -1\}^n$ the set of all inclination vectors
- $B : I \to I$ influence function $Bi$ - decision vector
- Power indices in voting literature.

# More investigations of the influence model

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- A dynamic model of influence based on aggregation functions.

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- Aim of the paper:
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  - to provide a general analysis of convergence in the aggregation model and to give more practical conditions based on influential players.

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- Generalization to a *stochastic influence function* - An influence function $B$ can be coded by a $2^n \times 2^n$ row-stochastic matrix $\mathbf{B} = [b_{S,T}]_{S,T \subseteq N}$, where an entry $b_{S,T}$ is the probability that $B(S)$ is $T$:

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  2. If $\mathbf{x} \leq \mathbf{x}'$ then $A(\mathbf{x}) \leq A(\mathbf{x}')$ (nondecreasingness).
- Numerous examples: all kinds of means (geometric, harmonic, quasi-arithmetic) and their weighted version, any combination of minimum and maximum (lattice polynomials or Sugeno integrals), Choquet integrals, triangular norms, copulas, etc.

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  which indicates the probability of each agent to say 'yes' after influence.
- Considering that these probabilities are independent among agents, we find that the probability of transition from the yes-coalition $S$ to the yes-coalition $T$ is

  $$b_{S,T} = \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i), \quad \forall S, T \subseteq N,$$

  which determines $\mathbf{B}$.

Definition

1. Let $A_i$ be the aggregation function of agent $i$. Agent $j \in N$ is *influential in $A_i$* if $A_i(1_j) > 0$ and $A_i(1_{N \setminus j}) < 1$.
2. The *graph of influence* is a directed graph $G_{A_1,\ldots,A_n} = (N, E)$ whose set of nodes is $N$, and there is an arc $(i, j)$ from $i$ to $j$ if $i$ is influential in $A_j$.
   We denote its undirected version by $G^0_{A_1,\ldots,A_n}$.

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  4. There are no other terminal states than the trivial terminal classes if the undirected graph $G^0_{A_1,\ldots,A_n}$ is connected.

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  3. or collections $\mathcal{C}$ of nonempty sets with the property that $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p$, where each subcollection $\mathcal{C}_j$ is a Boolean lattice $[S_j, S_j \cup K_j]$, $S_j \neq \emptyset$, $S_j \cup K_j \neq N$, and at least one $K_j$ is nonempty.

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- We call *cyclic terminal classes* those terminal classes of the second type and *regular terminal classes* those of the third type. Regular terminal classes can be periodic.

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