Signaling games and Independence-Friendly Logic

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- Failure of determinacy (bivalence)

Failure of bivalence

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- The sentence φ_{inf}

$$\exists w \forall x(\exists y/\{w\})(\exists z/\{w, x\})(x = z \wedge w \neq y)$$

which defines (Dedekind) infinity is undeterminate on all finite structures

Beyond Undeterminacy: the methodology

- Suggestion by Aitaj (Blass and Gurevich, 1986)
- M. Sevenster (2006), doctoral dissertation (ILLC)
- M. Sevenster and G. Sandu (2010), Equilibrium semantics of languages with perfect information, *APAL*
- A. Mann & G. Sandu & M. Sevenster, 2011, *Independence-Friendly Logic: A Game-theoretic Approach*, CUP

Beyond Undeterminacy: the methodology

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- Truth in a structure $\Longrightarrow$ probabilistic value in a structure
- The probabilistic value in a structure is Eloise's expected utility

Example: Matching Pennies

- The strategic game in a two element structure:

	a_1	a_2
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  is an equilibrium in the game.
- Eloise's expected utility is $\frac{1}{2}$.

Example: Matching Pennies

- The probabilistic values of Matching Pennies and the Inverted Matching Pennies:

Cardinality of M	φ_{MP}	φ_{IMP}
1	1	0
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n-1}{n}$

Example: Infinity

- The probabilistic value of φ_{inf}

$$\exists w \forall x (\exists y/\{w\})(\exists z/\{w,x\})(x = z \wedge w \neq y)$$

on a structure M of cardinality n is $\frac{n-1}{n}$.

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  on a structure M of cardinality n is $\frac{n-1}{n}$.
- Thus as M grows to infinity, the probabilistic value of φ_{inf} approaches 1.
- The probabilistic values of φ_{inf} and $\varphi_{IMP} = \forall x (\exists y / \{x\}) x \neq y$ coincide on all finite structures.

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- Lewis argues that a word acquires its meaning in virtue of its role in the solution to various signaling problems.

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- C employs an encoding $f : S \longrightarrow \Sigma$ to choose a signal for every situation.
- A employs a function $g : \Sigma \longrightarrow R$ to decide which action to perform in response to the signal it receives.
- A *signaling system* is a pair (f, g) of encoding and decoding functions such that their composition $g \bullet f = b$.

Example

A driver who is trying to back into a parking space. She has an assistant who gets out of the car and stands in a location where she can simultaneously see how much space there is behind the car and be seen by the driver. There are two states of affairs the assistant wishes to communicate, i.e., whether there is enough space behind the car for the driver to continue to back up. The assistant has two signals at her disposal: she can stands palms facing in or palms facing out. The driver has two possible responses: she can back up or she can stop.
There are two solutions (signaling systems) for this signaling problem.

Signaling systems in IF logic

- (Hodges') IF sentence $\forall x \exists z (\exists y/\{x\}) x = y$ can be modified to express a Lewisian signaling system:

$$\forall x \exists z (\exists y/\{x\})[S(x) \rightarrow (\Sigma(z) \wedge R(y) \wedge y = b(x))].$$

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- We slightly modify Lewis' signaling games and take A's taks to decode the situation the message was sent from:

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- We consider structures

$$M = (D, S^M, \Sigma^M, R^M)$$

such that $D = \{s_1, ..., s_n, t_1, ..., t_m\}$,
$S^M = R^M = \{s_1, ..., s_n\}, \Sigma^M = \{t_1, ..., t_m\}$.

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- In this case the $M \models^{+} \varphi_{sig}$ and Eloise's winning strategy forms a signaling system.
- B. Skirms asked: What happens when $m < n$?

Signaling systems in IF logic

Proposition

Let m, n be natural numbers such that $0 \leq m < n$ and M be a finite structure

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such that

$$D = \{s_1, ..., s_n, t_1, ..., t_m\}$$
$$S^M = R^M = \{s_1, ..., s_n\}$$
$$\Sigma^M = \{t_1, ..., t_m\}$$

Then $M \not\models^+ \varphi_{sig}$ and $M \not\models^- \varphi_{sig}$.

Signaling systems in IF logic

Proposition

Let $0 \leq m < n$. The probabilistic value of φ_{sig} on a structure $M = (D, S^M, \Sigma^M, R^M)$ with n states and m signals is $\frac{m}{n}$.

Signaling systems in IF logic

Theorem

There exists an IF sentence φ *such that for every integers* m, n, $0 \leq m < n$, *there is a structure* $M = (D, S^M, \Sigma^M, R^M)$ *such that the value of* φ *in* M *is* $\frac{m}{n}$.

Signaling systems in IF logic

Theorem

There exists an IF sentence φ such that for every integers m, n, $0 \leq m < n$, there is a structure $M = (D, S^M, \Sigma^M, R^M)$ such that the value of φ in M is $\frac{m}{n}$.

- This result is to be compared with:

Theorem

(Sevenster & Sandu, Galiani) Let $0 \leq m < n$ be integers and $q = \frac{m}{n}$. There exists an IF sentence that has value q on every structure.

Further applications of equilibrium semantics

- Monty Hall (A. Mann)
- Sleeping Beauty (A.Mann & V. Aarnio)