Logic and Infinite Games: Some Results and Perspectives

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LogICCC Final Conference, Berlin, September 2011

Background

APPLICATION OF RECURSIVE ARITHMETIC TO THE PROBLEM OF CIRCUIT SYNTHESIS

Alonzo Church

RESTRICTED RECURSIVE ARITHMETIC

Primitive symbols are individual (i.e., numerical) variables x, y, z, t,..., singulary functional constants $i_1, i_2, \ldots, i_\mu$, the individual constant 0, the accent ' as a notation for <u>successor</u> (of a number), the notation ( ) for application of a singulary function to its argument, connectives of the propositional calculus, and brackets [ ].

Axioms are all tautologous wffs. Rules are modus ponens; substitution for individual variables; mathematical induction,

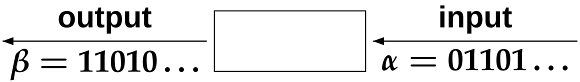
$$\text{from } P \supset S^{a}_{a'}P| \text{ and } S^{a}_{0}P| \text{ to infer } P;$$

and any one of several alternative recursion schemata or sets of recursion schemata.

Alonzo Church

Church's Problem

Given a requirement on a bit stream transformation



fill the box by a machine with output, satisfying the requirement (or state that the requirement is not satisfiable).

An important concrete case:

- **Requirements are formulated in MSO-logic.**
- **The machine should be a finite automaton.**

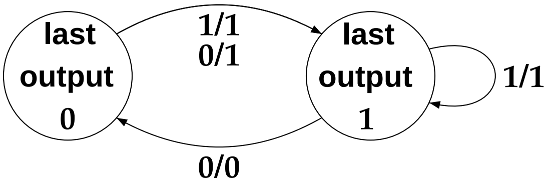
Helpful perspective: Infinite two-person game.

Example

Requirement:

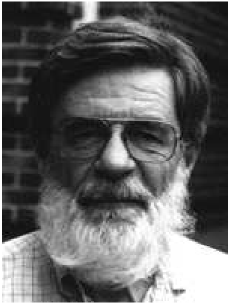
1. $\forall t : \alpha(t) = 1 \;\; \rightarrow \;\; \beta(t) = 1$
2. $\neg \exists t : \beta(t) = \beta(t+1) = 0$
3. $\exists^{\omega} t \; \alpha(t) = 0 \;\; \rightarrow \;\; \exists^{\omega} \; t \; \beta(t) = 0$

A solution:



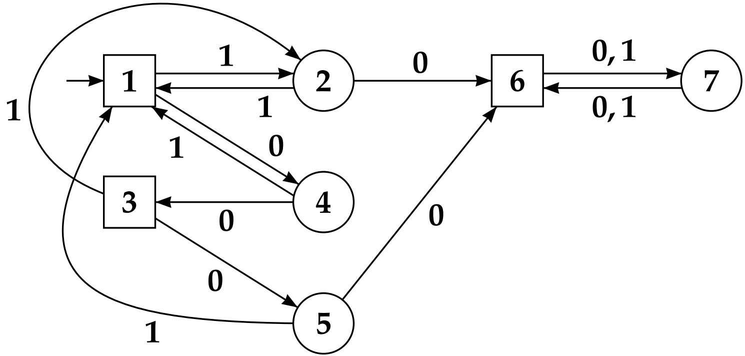
Step 1: From Formula to Game on Graph

J.R. Büchi



R. McNaughton

Example again



Player 2 wins if the infinitely often visited states are:
$\{1, 2, 3, 4\}$ **or** $\{1, 2, 3, 4, 5\}$ **or** $\{1, 3, 4, 5\}$ **or** $\{1, 4\}$

This is a Muller game (standard form of regular infinite games).

Step 2: Büchi-Landweber Theorem (1969)

For a Muller game

and hence for each MSO-requirement:

1. **either Player 1 or Player 2 has a winning strategy,**
2. **it is decidable who wins,**
3. **and a finite-state winning strategy for the respective winner is computable.**

Popular approach today goes via “parity games”.

Trends in the Theory of Infinite Games

- **Generalizations of the game model:**

  Infinite-state, concurrent, stochastic, timed, weighted, distributed, multi-player games

- **In this talk:**
  Construction of “good” strategies

A strategy may be called “good” if . . .

- it is generous
  i.e., discloses moves in advance to the opponent,
- it is simple
  i.e., is definable with weak logical means,
- it is responsive
  i.e., serves requests quickly,
- it is permissive
  i.e., allows several choices (nondeterministically),
- it is finite-time
  i.e., pursues just a safety objective and wins infinite plays in finite time.

Strategies with Lookahead

The Idea

If a player discloses at time i his moves for times $i+1, \ldots, f(i)$ then the opponent has a corresponding lookahead.

Question:

If Player 2 wins the standard game, for which rates f of generosity can he still win?
At which rates will the game be won by the opponent?

Recall: $F : \alpha \mapsto \beta$ is continuous (in the Cantor topology over the space of infinite sequences) if

$\beta(i)$ is determined by a finite prefix $\alpha(0) \ldots \alpha(j)$ of α

Strategies with given lookahead rate are just uniformly continuous functions.

Two Results

A Collapse Result:

In a regular infinite game, a player wins with any lookahead f iff he wins with a constant lookahead $g(i) = i + k$, and the minimal such k can be computed.

So the possibility to win with lookahead can be decided.

A complementary result:

For context-free infinite games,
the possibility to win with lookahead is undecidable,
and in this case the lookahead cannot be bounded by any elementary function.

(Fridman, Holtmann, Kaiser, Löding, Th., Zimmermann 2009/10)

Definability of Strategies

Strategies and Definability

A strategy for Player 2 is a map

$$\binom{\alpha(0)}{\beta(0)} \binom{\alpha(1)}{\beta(1)} \ldots \binom{\alpha(k)}{*} \quad \mapsto \quad 0/1$$

Strategies can be defined by sentences interpreted over such finite play prefixes,

in the sense that the truth value is the bit to be chosen.

The Büchi-Landweber Theorem says:

MSO-definable games can be solved with MSO-definable strategies.

General Problem: Relate the logic for describing winning strategies to the logic used to define the game.

Some Results

- Games definable in FO-logic (over $(\mathbb{N}, <)$) can be solved with strategies definable in the same logic.
- This fails for FO-logic over $(\mathbb{N}, +1)$.
- It also fails for the levels $B(\Sigma_n)$ of the quantifier alternation hierarchy within FO-logic over $(\mathbb{N}, <)$:
- Winning strategies need an increase of quantifier alternation rank by at least 1, and an increase by 2 suffices.
- For games defined in Presburger arithmetic, even full first-order arithmetic is insufficient for defining winning strategies.

(Chaturvedi, Olschewski, A. Rabinovich, Th. 2007/2011)

Responsive and Permissive Strategies

Request-Response Games

Format of winning condition:

$$\bigwedge_{i=1}^{k} \forall s(Rqu_i(s) \rightarrow \exists t\ (s < t \wedge Rsp_i(t)))$$

Measure the quality of a winning strategy in terms of the waiting times it induces.

- **Linear penalty model:**
  For each moment of waiting (for each condition) pay 1 unit

- **Quadratic penalty model:**
  For the i-th moment of waiting pay i units

Penalties of Plays and Strategies

- **Penalty of play prefix $\varrho(0)\ldots\varrho(n)$ is the sum of penalties so far, divided by number of “activations”.**
- **Penalty of play ϱ is the limes superior of the play prefix penalties.**
- **Penalty value of strategy σ is the supremum over the penalties of all plays compatible with σ.**
- **Call σ optimal if there is no other strategy with smaller penalty value.**

Results

- For the linear penalty model, a finite-state optimal strategy does not exist in general.

- For the quadratic penalty function
  one can decide whether a request-response game is won by Player 2 and in this case one can compute a finite-state optimal winning strategy.

- This can be generalized to games where a “response” involves several tasks to be satisfied according to given partial orders.

(Horn, Wallmeier, Th., Zimmermann 2008/2009)

Permissive Strategies

Model: Game on finite graph with the parity winning condition.

After a finite play prefix, a strategy just blocks certain edges.

Idea:

- **A strategy should narrow the system's behaviour as little as possible.**
- **This supports modular constructions: Adding requirements leads to a refinement of strategies.**

New Penalties

For strategy σ (of Player 2):

- **After a play prefix $\varrho(0)\ldots\varrho(n)$ it is the total number of edges blocked by σ up to time n.
  Call this $\pi_\sigma(\varrho(0)\ldots\varrho(n))$.**
- **Penalty of complete play ϱ is the lim sup over the average values $\frac{1}{n}\cdot\pi_\sigma(\varrho(0)\ldots\varrho(n))$**
- **Penalty associated with strategy σ:
  supremum of penalties of plays consistent with σ.**
- **Permissiveness value of game = infimum of this value over all winning strategies of Player 2.**

Result

- For parity games on finite graphs, the permissiveness value is computable.
- There are games where this value is not realizable by a finite-state strategy.

(Bouyer, Markey, Olschewski, Ummels 2010)

Open: Finite presentation of more general strategies such that computability of most permissive strategies is possible.

Winning Muller Games in Finite Time

Winning Games by Scores

Given a Muller game with the collection $\mathcal{F}$ of "winning loops" for Player 2
play this like a card game in the evening

... and of course go to sleep at some time.

Question: How can one terminate a play after finite time, declaring correctly the winner?

McNaughton's approach: Count for each loop F how often the loop F (as a set) was completely traversed without interruptions.

At time i denote this repetition number as $\mathrm{score}_F(i)$.

Winning = Reaching Score 3

- A player wins a Muller game iff he has a strategy which guarantees that score 3 is reached for one of his winning loops.
- This an be achieved by just ensuring that the opponent's scores stay ≤ 2.
- This amounts to solve a "safety game".

(Fearnley, Neider, R. Rabinovich, Zimmermann (2010/11))

We obtain a new approach to strategy construction — and to strategy optimization.

Conclusion

Church's Problem is far from closed:

- **Even for the (classical) infinite two-person games, we have not yet understood completely how to construct strategies that are "good" — and it is even less clear how to handle multiple optimization criteria.**
- **For the connection between games and logic, a central question is to better understand the relation between definability of games and strategies.**
- **In particular:**
  Is there a compositional framework of strategy construction which reflects the structure of the (logical) specifications and works without the detour through automata theory (algorithmic theory of labelled graphs)?