Characteristic classes associated to Q-bundles

Alexei Kotov

Institute of Mathematics
University of Luxembourg

ESF Mathematics Conference
"Algebraic Aspects in Geometry
Bedlewo, October 2007"

Plan of the talk (joint with T.Strobl)

1. Q-manifolds. Morphisms and gauge symmetries.
2. Q-bundles and characteristic classes.
3. Examples (inspired by sigma models).

Gelfand-Neimark and Serre-Swann correspondence

Smooth manifold M	Commutative C^*-algebra $A = C^\infty(M)$
Submanifolds of M	Ideals of A
Smooth maps	Morphisms of commutative algebras $\Phi(fg) = \Phi(f)\Phi(g)$
Vector fields X on M	Derivatives of A $X(fg) = X(f)g + f\,X(g)$
Vector bundles on M	Projective modules of A

Graded manifolds

Graded commutative manifold	Graded commutative algebra: ▶ $A = \oplus_i A^i$, ▶ $A^i A^j \subset A^{i+j}$, ▶ $fg = (-1)^{ij} gf$ for each $f \in A^i$, $g \in A^j$
Graded smooth maps	Morphisms of graded commutative algebras $\Phi(fg) = \Phi(f)\Phi(g)$
Vector fields X on M of degree k	Graded derivatives of degree k: $X : A^i \rightarrow A^{i+k}$, such that $X(fg) = X(f)g + (-1)^{kj} f\, X(g)$, for each $f \in A$, $g \in A^j$

Graded manifolds

As it follows from the definition of a graded commutative algebra, A^0 is commutative subalgebra of A and A^i are A^0- modules.

Basic example: $V = \oplus_k V^k$ is a $\mathbb{Z}-$graded vector space;

$$\mathcal{A} = C^\infty(V^0) \otimes S^*(V^+ \oplus V^-)\,,$$

where

$$S^*(W) = T^*(W)/\langle a \otimes b - (-1)^{\deg(a)\deg(b)} b \otimes a\rangle$$

for each graded vector space W and $V^\pm = \oplus_{\pm k>0} V^k$.

A graded manifold $\mathcal{M}$ is a locally ringed space $(M, \mathcal{A})$ where M is a topological space and $\mathcal{A}$ is a sheaf of graded commutative algebras, such that M admits a cover by a countable set of coordinate charts the restriction of $\mathcal{A}$ to which is isomorphic to the basic example. The Euler field is the unique 0-th order derivative ϵ defined by the property $\epsilon(f) = jf$ for each $f \in \mathcal{A}^j$.

Graded manifolds

$\mathcal{M}$ is a graded manifold of degree p, iff $\mathcal{A}$ generated by $\mathcal{A}^i$, $i \leq k$

Example: $E[p]$ - a graded manifold of degree p

Let $\mathrm{p} : E \to M$ be a vector bundle of rank k, $\{U_\alpha\}$ an open cover of M, and $\mathrm{p}^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^k$ a trivialization with base coordinates x^μ and fiber coordinates η^a.

We put $\deg x^\mu = 0$, $\deg \eta^a = p$. The Euler field is

$$\epsilon = p\eta^a \frac{\partial}{\partial \eta^a} \,.$$

Transition cocycle is linear in fiber coordinates, therefore it preserves ϵ.

$$\mathcal{A} \simeq \begin{cases} \Gamma(M, \Lambda^* E^*)\,, & p \in 2\mathbb{Z}+1 \\ \Gamma(M, S^* E^*)\,, & p \in 2\mathbb{Z}\,. \end{cases}$$

Q-MANIFOLDS

A Q-manifold (Q-manifold of degree p) is a graded manifold (graded manifold of degree p) endowed with a vector field Q of degree 1 which obeys the Master equation $Q^2 = 0$.

A morphism of $Q-$ manifolds is a degree preserving map such that the push-forward of the $Q-$ field on the source graded manifold is well-defined and equals to the $Q-$ field on the target manifold.

More precisely, given a morphism of graded manifolds $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$, the following chain property holds:

$$Q_1 \circ \Phi^* - \Phi^* \circ Q_2 = 0\,,$$

where $\Phi^* : \mathcal{A}_2 \to \mathcal{A}_1$ is the pull-back map acting on functions.

L_∞ ALGEBROIDS AS Q_1-MANIFOLDS

There is a one-to-one correspondence between Q_1 manifolds and Lie algebroids $(E, \rho, [,])$. The morphism of Lie algebroids is the morphism of the corresponding $Q-$ manifolds.

Given a vector bundle $E \to M$ and a Q-structure on $E[1]$, one has

$$\mathcal{A}^0 = C^\infty(M) \xrightarrow{Q} \mathcal{A}^1 = \Gamma(E^*) \xrightarrow{Q} \mathcal{A}^2 = \Gamma(\Lambda^2 E^*)\,.$$

We define:

- $\rho(s)f\rangle = \langle Qf, s\rangle$, $f \in C^\infty(M)$, $s \in \Gamma(E)$
- $\langle \alpha, [s_1, s_2]\rangle = \langle Q\alpha, s_1 \wedge s_2\rangle - \rho(s_1)\langle \alpha, s_2\rangle + \rho(s_2)\langle \alpha, s_1\rangle$, $\alpha \in \Gamma(E^*)$, $s_i \in \Gamma(E)$.

Derived brackets

A graded Lie algebra $\mathfrak{g}$ is a graded vector space $\mathfrak{g} = \oplus_i \mathfrak{g}^i$ together with a bilinear operation of degree 0 $[,] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ which satisfies the graded Jacobi condition:

$$[a,[b,c]] = [[a,b],c] + (-1)^{ij}[a,[b,c]]\,,$$

where $a \in \mathfrak{g}^i$, $b \in \mathfrak{g}^j$.

Example Vector fields on a graded manifold with respect to the supercommutator $[X_1, X_2] = X_1X_2 - (-1)^{k_1k_2}X_2X_1$, where X_i is a derivative of degree k_i.

A differential graded Lie algebra $\mathfrak{g}$ is a graded Lie algebra together with a differential δ of degree 1, which satisfies the compatibility condition: $\delta[a,b] = [\delta a, b] + (-1)^i[a, \delta b]$, where $a \in \mathfrak{g}^i$.

Example Vector fields on a Q-manifold, $\delta = ad_Q$: $ad_Q^2 = 0$

Derived brackets

Let $(\mathfrak{g}, \delta)$ be a differential graded Lie algebra. Define the derived bracket

$$[a, b]'_\delta := [a, [\delta, b]]$$

If $\mathfrak{h}$ is a graded commutative subalgebra of $\mathfrak{g}$, then $(\mathfrak{h}, [,]_\delta)$ is graded Lie algebra with a shift of grading.

Example g the Lie algebra of vector fields on a Q_1 manifold, $\mathfrak{h} = \Gamma(M, E)$ acting by contractions. The induced derived bracket defines the corresponding Lie algebroid structure.

In particular, $E = TM$, $\mathcal{A} = \Omega(M)$, and Q is the de Rham operator d. Then the derived bracket simply reproduces the known formula:

$$i_{[X,Y]} = [i_X, [\mathrm{d}, i_Y]] .$$

Examples

1. Graded Lie algebroid, iff $Q_m = 0,\ m \neq 1$: Q_1 defines a graded Lie algebroid bracket
2. Differential graded Lie algebroid, iff $Q_m = 0,\ m > 1$: Q_1 defines a graded Lie algebroid bracket, Q_0 a $C^\infty(M)-$linear (fiber) differential compatible with the graded Lie algebroid structure
3. Differential graded Lie 2-algebroid, iff $Q_m = 0,\ m > 2$

We start with a simple example of $\mathfrak{g}-$valued 1-forms on a smooth manifold M, where $\mathfrak{g}$ is a Lie algebra. Given $A \in \Omega^1(M, \mathfrak{g})$, interpreted as a connection in a trivial bundle $M \times G$, Lie(G)$= \mathfrak{g}$, we look at its curvature:

$$F_A := \mathrm{d}A + \frac{1}{2}[A, A] \,.$$

The group of $G-$valued functions on M is acting on connections by:

$$A^g = g^{-1}\mathrm{d}g + \mathrm{Ad}_{g^{-1}}(A) \,,$$

where $g^{-1}\mathrm{d}g$ is the pull-back by g of the (left) Maurer-Cartan form on the Lie group and Ad is the adjoint action.

The infinitesimal version is governed by a $\mathfrak{g}-$valued function :

$$\delta_\epsilon A := \frac{\mathrm{d}}{\mathrm{d}t} A^{\exp(t\epsilon)} \mid_{t=0} = \mathrm{d}\epsilon + [A, \epsilon] \,.$$

The condition of flatness, $F_A = 0$, can be also regarded as Maurer-Cartan equation for A.

Let us adapt this example to the language of dg or Q-manifolds. As we already know, a Lie algebra can be treated as a Q-manifold $\mathfrak{g}[1]$, such that the algebra of functions becomes isomorphic to $\Lambda(\mathfrak{g}^*)$ with the Q-field given by the Chevalley-Eilenberg differential:

$$\mathrm{d}(\alpha)(\eta, \eta') = -\alpha([\eta, \eta'])\,, \quad \alpha \in \mathfrak{g}^*\,, \eta, \eta' \in \mathfrak{g}$$

or equivalently (in a chosen basis): $\mathrm{d}\xi^i = -\frac{1}{2} C^i_{jk} \xi^j \wedge \xi^k$.

Q-manifolds. Morphisms, gauge symmetries.

The product of $T[1]M$ and $\mathfrak{g}[1]$ is again a Q-manifold, the Q-structure of which is given by the sum of de Rham and Chevalley-Eilenberg derivations extended to the product in the standard way. A $\mathfrak{g}-$valued 1-form on M can be thought of as a degree preserving map $\varphi \colon T[1]M \to \mathfrak{g}[1]$ and its graph as a section of the bundle

$$T[1]M \times \mathfrak{g}[1] \to T[1]M \,. \tag{1}$$

The pull-back of φ is acting as follows: for each $\omega \in \Omega(M)$, $\alpha \in \Lambda^p(\mathfrak{g}^*)$ one has

$$\varphi^*(\alpha \otimes \omega) = \alpha(\underbrace{A \overset{\wedge}{,} \ \ldots \ \overset{\wedge}{,} A}_{p\,\text{times}}) \wedge \omega \,.$$

Any $\mathfrak{g}-$valued function ϵ, acting by the contraction ι_ϵ on $\Omega(M) \otimes \Lambda(\mathfrak{g}^*)$, can be considered as a super-derivation of degree -1, which super-commutes with $\Omega(M)$. The last property implies that it can be identified with a vertical vector field on the total space of (1). The following identity holds $\forall \omega \in \Omega(M), \alpha \in \Lambda^p(\mathfrak{g}^*)$:

$$(\mathrm{d}\,\varphi^* - \varphi^*\,(\mathrm{d} + \mathrm{d}_{\mathfrak{g}}))\;(\alpha \otimes \omega) =$$

$$\sum_k (-1)^{k+1} \alpha(\,A \overset{\wedge}{,}\;\ldots \overset{\wedge}{,} \overbrace{F_A}^{k} \overset{\wedge}{,} \ldots \overset{\wedge}{,} A\,) \wedge \omega,$$

$$\varphi^* L_\epsilon\,(\alpha \otimes \omega) = \sum_k \alpha(\,A, \overset{\wedge}{,}\;\ldots \overset{\wedge}{,} \overbrace{\delta_\epsilon A}^{k} \overset{\wedge}{,} \ldots \overset{\wedge}{,} A\,) \wedge \omega\;,$$

where $L_\epsilon = [Q, \iota_\epsilon]$, $\delta_\epsilon A$ is the gauge transformation defined above.

Note that:

- The bundle map $T[1]M \times \mathfrak{g}[1] \to T[1]M$ is a Q-morphism.
- Instead of thinking of an infinitesimal gauge transformation as a flow on the space of connections, we define a vector field on the total space of a bundle of Q-manifolds , the action of which on the space of connections, regarded as sections of the bundle, can be naturally induced.
- The curvature F_A is the only obstruction for A to be a Q-morphism.
- A is extended as a morphism of graded manifolds, but its infinitesimal variation $\delta_\epsilon A$ is extended (by the Leibnitz rule) as a derivation covering A.

A general fact: the space of infinitesimal variations (the tangent space) of a smooth map $\psi \colon M \to N$ can be identified with the space of sections of the pullback bundle $\psi^*(TN)$ or, equivalently, with the space of derivations $\delta \colon C^\infty(N) \to C^\infty(M)$ covering ψ:

$$\delta(hh') = \delta(h)\psi^*(h') + (-1)^{\deg(\delta)\deg(h)}\psi^*(h)\delta(h')$$

for any $h, h' \in C^\infty(N)$.

For each morphism (in the category of graded manifolds) of Q-manifolds $\psi \colon \mathcal{M}_1 \to \mathcal{M}_2$, the obstruction for ψ to be a Q-morphism

$$F := Q_1\psi^* - \psi^* Q_2$$

is a derivation of degree 1 covering ψ or a section of $\psi^* T[1]\mathcal{M}_2$.

Let $\pi \colon P \to M$ be a principal G-bundle. The local description is:

- A local cover $\{U_\alpha\}$ of M;
- A trivialization $\pi^{-1}U_\alpha \simeq U_\alpha \times G$;
- A transition cocycle $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ which acts on fibers by left multiplication.

The Atiyah algebroid $A(P)$ of π is TP/G, such that the space of sections is $\Gamma(TP)^G$, the bracket is the Lie bracket on G-invariant vector fields and the anchor map ρ is induced by π_*.

- The trivialization is $\rho^{-1}TU_\alpha \simeq TU_\alpha \times \mathfrak{g}$ where $\mathfrak{g}$ is identified with left invariant vector fields on G;
- The transition cocycle is $g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} + Ad_{g_{\alpha\beta}^{-1}}$ consisting of gauge transformations.

Q-manifolds. Morphisms, gauge symmetries.

- An Atiyah algebroid is a locally trivial bundle of Q-manifolds over $T[1]M$ with a typical fiber $\mathfrak{g}[1]$ glued by a cocycle which consists of gauge transformations:

$$\text{Atiyah algebroid}: A(P)[1] \to T[1]M .$$

- A connection in π, which is nothing but a lift of vector fields on M to G-invariant on the total space, is a section of this bundle in the category of graded manifolds:

$$\text{G} - \text{connection}: T[1]M \to A(P)[1] .$$

Let us examine once more the trivial example, which is a brick underlying a global design. Suppose $\mathcal{M} = \mathcal{N} \times \mathcal{F}$ is a product of two Q-manifolds $\mathcal{N}$ and $\mathcal{F}$ and $\pi\colon \mathcal{M} \to \mathcal{N}$ is a bundle given by the projection to the first factor.

Proposition Let $\mathcal{G}$ be a graded Lie subalgebra of vector fields on $\mathcal{F}$, closed under the derived bracket. Then the space of functions on $\mathcal{N}$ taking values in $\mathcal{G}$ is a Lie subalgebra of vertical vector fields closed under the derived bracket on the total space $\mathcal{N} \times \mathcal{F}$.

Let us use the notation for the following Lie algebra of vector fields on the total space:

$$\mathcal{G}' := ad_Q\left(C^\infty(\mathcal{N}, \mathcal{G})\right) \cap \mathcal{D}^0(\mathcal{N} \times \mathcal{F})\,.$$

Is is not a surprise for us that $\mathcal{G}'$ consists of vertical vector fields. Indeed, suppose we are given $X \in \mathcal{G}'$, then there exists some element $\epsilon \in \mathcal{G}$ such that $X = [Q, \epsilon]$. Both of two vector fields in the commutator are $\pi-$projectable, since $\pi_*(Q) = Q_1$ and $\pi_*(\epsilon) = 0$, thus $\pi_*(X) = 0$.

It well-known that exponentiating a vertical vector field (at least locally), we obtain a fiber-wisely acting automorphism, i.e. an automorphism Ψ satisfying $\pi \circ \Psi = \pi$. Apparently, the set of fiber-wisely acting automorphisms is a subgroup of all automorphisms of a bundle and a composition of Ψ with any section of π is again a section. In this way we can now return to the general, nontrivial bundle situation, formulating the following:

Q-BUNDLES

Definition A Q-bundle $\pi\colon \mathcal{M} \to \mathcal{M}_1$ with typical fiber $\mathcal{F}$ and a holonomy algebra $\mathcal{G} \subset \mathcal{D}^{<0}(\mathcal{F})$ (a chosen graded Lie subalgebra of vector fields on $\mathcal{F}$, closed under the derived bracket) is a surjective Q-morphism, satisfying the local triviality condition: there exists an open cover $\{\mathcal{U}_i\}$ of $\mathcal{M}_1$ such that the restriction of π to each $\mathcal{U}_i$ admits a trivialization $\pi^{-1}(\mathcal{U}_i) \simeq \mathcal{U}_i \times \mathcal{F}$ in the category of Q-manifolds and this trivialization is glued over $\mathcal{U}_i \cap \mathcal{U}_j$ by inner automorphisms which belong to $\exp(\mathcal{G}')$ where $\mathcal{G}'$ is as above with $\mathcal{N} = \mathcal{U}_i \cap \mathcal{U}_j$.

A gauge field is a section of π in the category of graded manifolds, that is, a degree preserving map $\varphi\colon \mathcal{M}_1 \to \mathcal{M}$ which obeys $\pi \circ \varphi = \mathrm{Id}$. A gauge transformation (an infinitesimal gauge transformation) is a fiber-wisely acting inner automorphism (vertical inner derivation) of the total space of π.

Question:

What is a meaning of the Chern-Weil formalism $S(\mathfrak{g}^*)^G \to H^{\mathrm{even}}(M,\mathbb{R})$ in terms of $Q-$bundles?

For a principal bundle we choose a connection A with the curvature F_A and a G-invariant polynomial $\chi \in S^r(\mathfrak{g}^*)^G$, then obtain:

$$\chi(F_A \overset{\wedge}{,} \ldots \overset{\wedge}{,} F_A) \in \Omega^{2r}_{cl}(M)$$

and take its cohomology class which does not depend on the choice of connections.

Characteristic classes of Q-bundles

The operator

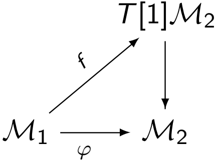
$$F := Q_1\varphi^* - \varphi^* Q_2 \,, \tag{2}$$

called *the field strength*, being a replacement of the curvature, is a degree one derivation of functions on the target manifold $\mathcal{M}_2$ taking values in functions in the source manifold $\mathcal{M}_1$ and covering φ^*.

$$\begin{array}{ccc} T[1]\mathcal{M}_1 & \xrightarrow{\varphi_*} & T[1]\mathcal{M}_2 \\ \uparrow Q_1 & & \uparrow Q_2 \\ \mathcal{M}_1 & \xrightarrow[\varphi]{} & \mathcal{M}_2 \end{array}$$

where the homological vector fields are considered as maps and, being of degree one, the tangent bundle was shifted in degree so that the maps are morphisms of graded manifolds.

Now one notes that both ways from $\mathcal{M}_1$ to $T[1]\mathcal{M}_2$ end in the same fiber over $\mathcal{M}_2$; thus it is meaningful to define the difference $f\colon \varphi_* \circ Q_1 - Q_2 \circ \varphi$, covering φ



It is easy to convince oneself that for any function $h \in C^\infty(\mathcal{M}_2)$ and any $\alpha, \beta \in C^\infty(T[1]\mathcal{M}_2)$ one has

$$f^*(h) = \varphi^*(h)\,, \quad f^*(\mathrm{d}h) = F(h)\,, \quad f^*(\alpha\beta) = f^*(\alpha) f^*(\beta)\,. \quad (3)$$

We shall see below that f is a Q-morphism, if $T[1]\mathcal{M}_2$ is endowed with a suitable Q-structure.

For any graded manifold $\mathcal{M}$, the algebra of functions on $T[1]\mathcal{M}$ admits a simple description as the algebra of super differential forms $\Omega(\mathcal{M})$ (according to the Bernstein-Leites sign convention). More precisely, the algebra of forms is generated by h and $\mathrm{d}h$ for all functions h with the following relations:

$$h\,\mathrm{d}h' = (-1)^{\deg(h)(\deg(h')+1)}\mathrm{d}h'\,h\,,$$
$$\mathrm{d}(hh') = \mathrm{d}h\,h' + (-1)^{\deg(h)}h\,\mathrm{d}h'\,.$$

A vector field X of degree p gives a contraction of degree $p-1$ acting as follows:

$$\iota_X(f\,\mathrm{d}h) = (-1)^{\deg(f)(\deg(X)+1)}f\,X(h)\,.$$

The super Lie derivative along X, an operator of degree p, is defined as the commutator

$$\mathcal{L}_X := \iota_X\mathrm{d} + (-1)^{\deg(X)}\mathrm{d}\iota_X\,.$$

By construction, $\mathcal{L}_X$ super-commutes with the de Rham differential and agrees with the action of vector fields on functions, $\mathcal{L}_X(f) = X(f)$. Furthermore, one can also check that the Lie derivative respects the super-Lie algebra of vector fields, generalizing the formulas for even manifolds, such that the following identities hold:

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \ , \quad [\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]} \ .$$

In particular, if Q is a homological vector field, we immediately obtain that

$$[\mathrm{d}, \mathcal{L}_Q] = [\mathcal{L}_Q, \mathcal{L}_Q] = 0 \ .$$

As a corollary we conclude that the total space of $T[1]\mathcal{M}$ for a Q-manifold $\mathcal{M}$ is a bi-graded manifold supplied with a couple of super-commuting Q-structures which are of degree one w.r.t. the first and the second gradings, respectively. Let us denote the total differential as $Q_{T\mathcal{M}} = \mathrm{d} + \mathcal{L}_Q$.

Proposition The map $f\colon \mathcal{M}_1 \to T[1]\mathcal{M}_2$ is a Q-morphism w.r.t. the total Q-structure on the target, that is, the following chain property holds:

$$Q_1 f^* - f^* Q_{T\mathcal{M}_2} = 0 \,. \quad (4)$$

A natural example of the chain map property of f is provided by the Weil algebra. It is well-known that, if one has a graded morphism from $\Lambda(\mathfrak{g}^*)$ of a Lie algebra $\mathfrak{g}$ to some differential graded commutative algebra $\mathcal{A}$, which is not necessarily a chain map, we can always extend it as a chain map, acting from the Weil algebra $W(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ to $\mathcal{A}$. The construction is working as follows: given a graded morphism $\Lambda(\mathfrak{g}^*) \to \mathcal{A}$, we identify it with some A which belongs to the dg Lie algebra $\mathcal{A} \otimes \mathfrak{g}$, where the differential and the bracket are extended by linearity:

$$\mathrm{d}(\alpha \otimes X) := \mathrm{d}\alpha \otimes X\,, \quad [\alpha \otimes X, \beta \otimes Y] := \alpha\beta \otimes [X, Y]\,,$$

where $\alpha, \beta \in \mathcal{A}$ and $X, Y \in \mathfrak{g}$.

Defining $F_A := \mathrm{d}A + \frac{1}{2}[A, A]$ (we recognize the curvature of a connection in a trivial bundle as a particular example), the required map $W(g) \to \mathcal{A}$ is

$$\Phi \otimes \omega \mapsto \Phi(\underbrace{F_A, \ldots, F_A}_{q \text{ times}})\,\omega(\underbrace{A, \ldots, A}_{p \text{ times}}), \quad \Phi \in S^q(\mathfrak{g}^*)\,, \omega \in \Lambda^p(\mathfrak{g}^*)\,.$$

One can easily check that the grading and differential in the Weil algebra are choosen is such a way that the Weil algebra becomes isomorphic to $\Omega(\mathfrak{g}[1])$ supplied with the above total differential. Furthermore, the chain map described above is nothing but our map f, if $\mathcal{M}_2 = \mathfrak{g}[1]$ and $\mathcal{A} = C^\infty(\mathcal{M}_1)$.

$S^*(\mathfrak{g}^*)^G$ can be thought of as differential forms on $\mathfrak{g}[1]$ annihilated by the action of $\mathfrak{g}$ by shifts and adjoint transformations.

Definition A differential form $\omega \in \Omega(\mathcal{F})$ is called a (generalized) $\mathcal{G}$-base form, if $\mathcal{L}_\epsilon(\omega) = 0 = \mathcal{L}_{ad_Q(\epsilon)}(\omega)$ for each $\epsilon \in \mathcal{G}$. We denote the space of $\mathcal{G}$-base forms as $\Omega(\mathcal{F})_\mathcal{G}$.

Theorem Let $\pi\colon \mathcal{M} \to \mathcal{N}$ be a Q-bundle with a typical fiber $\mathcal{F}$, a holonomy algebra $\mathcal{G}$, and φ a section of π (in the graded sense). Then there is a well-defined map in cohomology

$$H^p(\Omega(\mathcal{F})_\mathcal{G}, Q_{T\mathcal{F}}) \to H^p(C^\infty(\mathcal{N}), Q_\mathcal{N}) \ ,$$

which does not depend on homotopies of φ.

Theorem Let $(\mathcal{S}, \omega)$ be a symplectic Qp-manifold, $p \in \mathbb{N}+$, N a $(p+2)$-dimensional manifold with boundary $\partial N = \Sigma$, and φ a (degree preserving) map from $T[1]N$ to $\mathcal{S}$. Then

$$\int_N f^*\omega = S^{AKSZ}_{\Sigma,(cl)} \tag{5}$$

where $S^{AKSZ}_{\Sigma,(cl)}$ is the (classical part of the) topological sigma model on the $(p+1)$-dimensional Σ obtained by the AKSZ-method.

- For $p = 1$ this gives the Poisson sigma model;
- For $p = 2$ the Courant sigma model;
- The above holds for arbitrary dimensions.

Some applications

Given a Q-bundle the typical fiber $\mathcal{F}$ of which is a PQ-manifold and the holonomy algebra is a subalgebra of the Lie algebra of all hamiltonian vector fields of negative degree, we immediately obtain the canonical characteristic class provided by Theorem: the corresponding basic form on the fiber is simply the symplectic form.

For an arbitrary PQ-bundle (a Q-bundle with PQ-fibers) over a base $T[1]N$ for a smooth manifold N one has a straightforward generalization of the 2d Chern class which is an even (odd) cohomology class if p is odd (even), respectively.

In particular, for the Atiyah algebroid of a principal $G-$bundle for a G the Lie algebra of which is supplied with a non-degenerate invariant symmetric form, the corresponding PQ-bundle has a typical fiber $\mathfrak{g}[1]$ together with a symplectic form ω provided by the invariant metric. The canonical characteristic class is nothing but the 2d Chern class of the principal $G-$bundle and the Theorem simply sounds as the well-known local statement:

$$\text{“Second Chern form} = \mathrm{d}\,(\text{Chern}-\text{Simons form})'' \ .$$

For the case $p = 1$ the corresponding PQ-bundle covers a locally trivial Poisson fibration and the corresponding characteristic class gives the image in $H^3(M, \mathbb{R})$ of the Dixmier-Douady class of a certain gerbe.