# Infinite dimensional Lie algebras, deformations and geometry

Alice Fialowski
Eötvös Lorand University
Budapest, Hungary

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## Definitions

– Lie algebra cohomology

$$C^q(\mathcal{L};\mathcal{L}) = \mathrm{Hom}(\Lambda^q \mathcal{L}, \mathcal{L}) \quad \text{skew-symm., } q\text{-linear}$$

$$d_q : C^q(\mathcal{L};\mathcal{L}) \to C^{q+1}(\mathcal{L},\mathcal{L}),$$

$$(d_q \omega)(l_1, \dots, l_{q+1}) = \sum_{1 \le s < t \le q+1} (-1)^{s+t-1} \omega([l_s, l_t], l_1, \dots \hat{l}_s \dots \hat{l}_t \dots l_{q+1})$$

$$+ \sum_{1 \le s \le q+1} (-1)^s [l_s, \omega(l_1, \dots \hat{l}_s \dots, l_{q+1})].$$

$d_{q+1} \circ d_q = 0 \Rightarrow C^{\cdot}(\mathcal{L},\mathcal{L})$ complex.

$Z^q(\mathcal{L};\mathcal{L})$ = $q$-cocycles

$\omega$ coboundary if $\exists\ \eta \in C^{q-1}(\mathcal{L},\mathcal{L})$ s.t.

$$\omega(l_1, \dots, l_q) = (d_{q-1}\eta)(l_1, \dots, l_q).$$

$B^q(\mathcal{L};\mathcal{L})$ coboundaries $\subset Z^q(\mathcal{L};\mathcal{L})$

Quotient space : $H^q(\mathcal{L};\mathcal{L})$.

$\mathcal{L}$ Lie algebra with bracket $\mu_0$ over a field $\mathbb{K}$.

– Deformations

a) intuitive : one-parameter family $\mathcal{L}_t$ of lie algebras with

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where $\varphi_i$ are $\mathcal{L}$-valued 2 cochains, $\mathcal{L}_t$ a lie algebra for each $t \in \mathbb{K}$. Here $\mathcal{L} = \mathcal{L}_0$.

2 deformations $\mathcal{L}_t$ and $\mathcal{L}_t'$ are equivalent, if $\exists$ lie. automorphism $\hat{\varphi}_t = id + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$, $\psi_i$ linear$|_{\mathbb{K}}$ s.t.

$$\mu_t'(x,y) = \hat{\varphi}_t^{-1}(\mu_t(\hat{\varphi}_t(x), \hat{\varphi}_t(y))) \quad \text{for } x,y \in \mathcal{L}.$$

Jacobi identity for $\mathcal{L}_t \Rightarrow \varphi_1$ is a cocycle:

$$d_2\varphi_1 = 0 :$$

$$d_2(\varphi_1)(x,y,z) := \varphi_1([x,y],z) + \varphi_1([y,z],x) + \varphi_1([z,x],y)$$
$$- [x, \varphi_1(y,z)] - [y, \varphi_1(z,x)] - [z, \varphi_1(x,y)] = 0$$

If $\varphi_1$ vanishes identically, the first nonvanishing $\varphi_i$ will be a cocycle.

If $\mu_t'$ is equivalent to $\mu_t$ (with cochains $\varphi_i'$), then

$$\varphi_1' - \varphi_1 = d_1\psi_1$$

$\Rightarrow \forall$ equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L};\mathcal{L})$. Call this class the differential of the deformation.

The differential of a family which is equivalent to a trivial family will be the zero cohomology class.

b) <u>other point of view</u>

Consider $\mathcal{L}_t$ as a Lie algebra over $\mathbb{K}[[t]]$.
Allow more parameters, or in general, <u>a commutative algebra</u> over $\mathbb{K}$ with id <u>as base</u> of a deformation.

If $A$ is a comm. alg. $|\mathbb{K}$ char $0$ which admits an augmentation $\varepsilon : A \to \mathbb{K}$ $(\varepsilon(1_A) = 1)$, then the ideal $m_\varepsilon := \operatorname{Ker} \varepsilon$ is maximal & if $m$ is a maximal ideal of $A$ s.t. $A/m \cong \mathbb{K}$, we get an augmentation.

<u>Spec</u>. $A$ is finitely generated $\mathbb{K}$-alg. over an algebraically closed field $\mathbb{K}$, then $A/m \cong \mathbb{K}$ for $\forall$ max. ideal $m$. So there exists at least one augmentation & all maximal ideals come from augmentations

Fix an augmentation $\varepsilon$ of $A$, $m = \operatorname{Ker} \varepsilon$ max. id.

<u>Definition</u>. A <u>global deformation</u> $\lambda$ of $\mathcal{L}$ with base $(A, m)$ is a <u>Lie $A$-algebra</u> structure <u>on $A \otimes_{\mathbb{K}} \mathcal{L}$</u>, with $[\,,\,]_\lambda$ s.t.

$$\varepsilon \otimes id : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Specifically: for all $a, b \in A$, $x, y \in \mathcal{L}$,

<u>1)</u> $[a \otimes x, b \otimes y]_\lambda = (ab \otimes id)[1 \otimes x, 1 \otimes y]_\lambda$

<u>2)</u> $[\cdot, \cdot]_\lambda$ is skew-symm. & satisfies the Jacobi id.

<u>3)</u> $\varepsilon \otimes id([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

So it is enough to give the elements $[1 \otimes x, 1 \otimes y]_\lambda$.

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_i{}' a_i \otimes z_i$$

where $B = \{z_i\}_{i \in I}$ is a basis of $\mathcal{L}$, $a_i = a_i(x, y) \in m$.
($\Sigma'$ means finite sum.)

A deformation is <u>trivial</u> if $A \otimes_{\mathbb{K}} \mathcal{L}$ has the trivial str.:

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y].$$

<u>Two deformations</u> of a Lie algebra $\mathcal{L}$ with base $A$ are <u>equivalent</u>, if $\exists$ Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes id$.

A deformation is <u>local</u> (in algebraic sense) if $A$ is a local $\mathbb{K}$-algebra with unique max. ideal.

<u>Formal</u>, if $A$ is a complete local algebra$|_{\mathbb{K}}$:

$A = \varprojlim_{n \to \infty} (A/m^n)$, where $m$ is the maximal ideal of $A$.

<u>Infinitesimal</u>, if $m_A^2 = 0$.

<u>Ex.</u> <u>$A$ = algebra of regular functions of an affine variety $V$</u>: $A = \mathbb{K}[V]$. - coordinate algebra of $V$.

<u>Known</u>: $\forall$ finitely generated $\mathbb{K}$-algebra (as ring) which is reduced (no nilpotent elements) is the algebra of regular functions of a suitable affine variety. ($\mathbb{K}$ alg. closed)

$$m_x \text{ maximal ideals } \Leftrightarrow x \in V$$
$$\Leftrightarrow \varepsilon_x \text{ augmentations}$$

Fixing $\varepsilon_{x_0}$: fixing $x_0 \in V$.

If $A$ is a non-local ring, different maximal ideals.

Let $\mathcal{L}$ be a $\mathbb{K}$-vector space, assume $\exists [\cdot, \cdot]_A$ on $A \otimes_{\mathbb{K}} \mathcal{L}$. Given an augmentation $\varepsilon: A \to \mathbb{K}$ with $m_\varepsilon = \ker \varepsilon$, we get a Lie $\mathbb{K}$-algebra str. $\mathcal{L}^\varepsilon = (\mathcal{L}, [\cdot, \cdot]_\varepsilon)$ on $\mathcal{L}$ by passing to $A/m_\varepsilon$. This way the Lie algebra $A \otimes_{\mathbb{K}} \mathcal{L}$ gives a family of Lie algebra str-s on $\mathcal{L}$, parameterized by the points of $V$.

$$V \to \{\text{set of Lie alg. str-s on } \mathcal{L}\}$$

<u>$A \otimes_{\mathbb{K}} \mathcal{L}$ is a global deformation with basis $(A, m_x)$ for $\forall x \in V$.</u>

Special case: Deformations over the affine line $A^1$. The corresponding algebra: $A = \mathbb{K}[A^1] = \mathbb{K}[t]$, the algebra of polynomials in one variable.
For a deformation of the Lie algebra $\mathcal{L} = \mathcal{L}_0$ over the affine line, the Lie str. $\mathcal{L}_\alpha$ in the fiber over the point $\alpha \in \mathbb{K}$ is given by considering the augmentation corresponding to the maximal ideal $m_\alpha = (t-\alpha)$.

Call such deformations geometric.

$A = \mathbb{K}[V]$

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Rigidity

Given a family of Lie algebras containing $\mathcal{L}$ as a special element $\mathcal{L}_0$, any element $\mathcal{L}_t$ in the family "nearby" will be isomorphic to $\mathcal{L}_0$.

Definition a) $\mathcal{L}$ is infinitesimally rigid $\Leftrightarrow$ $\forall$ infinitesimal deformation of it is equivalent to the trivial one.

b) $\mathcal{L}$ is formally rigid $\Leftrightarrow$ every formal deformation of it is equivalent to the trivial one.

Remark: In these cases there is only one closed point, the point 0 itself, so every infinitesimal & formal deformation is already local. This is different on the geometric and analytic level (convergent power series). There locally equivalent means that there exists an open neighborhood $U$ of 0 s.t. the family restricted to it is equivalent to the trivial one $\Rightarrow$

$$\mathcal{L}_\alpha \cong \mathcal{L}_0 \quad \forall \alpha \in U.$$

Ex. $\mathcal{L}$ any nonabelian Lie alg., $[\,,\,]$. Define a family of Lie algebras $\mathcal{L}_t|_{\mathbb{K}[t]}$ (geometrically over the affine line $\mathbb{K}$) by having $\quad [x,y]_t := (1-t)[x,y]$.

If $t=0$, we get back $\mathcal{L}$. As long as $t\neq 1$, the algebras $\mathcal{L}_t$ are isomorphic to $\mathcal{L}$, but $\mathcal{L}_1$ is abelian so is not isomorphic. Hence such a family will never be a trivial family. But if we restrict the family to the Zariski open subset $\mathbb{K}\setminus\{1\}$ of $\mathbb{K}$, we get a trivial family.

$\mathbb{K}$ – algebraically closed, char $=0$ field

$A$ – finitely generated reduced $\mathbb{K}$-alg.

affine varieties with base $V$.

$W\subset V$ is Zariski-closed if it is the vanishing set of an ideal in $A=\mathbb{K}[V]$. $W$ is an affine variety.

A basis of the open sets of the topology: Zariski-open affine varieties. They are open in $V$ and are affine.

In particular, given an arbitrary Z-open subset $U$ containing the point $x\in V$ we can always find an open subset $W$ of $U$ which is affine.

Use Spec$(A)$ to denote the variety $V$ & identify the (closed) points in $V$ with the maximal ideals.

Definition. $\mathcal{L}$ is geometrically rigid if for every geometric deformation of $\mathcal{L}$ with base $(A,m)$, $\exists$ a Zariski open affine neighborhood Spec$(B)$ of the point $m$ in Spec$(A)$ s.t. the restricted family over $(B,m)$ is equivalent to the trivial one.

Ex. In the case if the base field is $\mathbb{C}$ or $\mathbb{R}$ & the base of the deformation is a finite-dimensional analytic manifold $M$ with a chosen point $x_0 \in M$, we consider also the usual topology on $M$. In particular, we can talk about open subsets of $M$ containing $x_0$.

Definition. $\mathcal{L}|_{\mathbb{C},\mathbb{R}}$ is analytically rigid, if for every family over a finite-dimensional analytic manifold $(M, x_0)$ with base point $x_0$, with special fiber $\mathcal{L} \cong \mathcal{L}_{x_0}$ over the point $x_0 \in M$, there is an open neighborhood $U$ of $x_0$, s.t. the restriction of this family is equivalent to the trivial family.

As the analytic rigidity is a (geometrically) local condition, it is enough to establish rigidity by considering families over $\mathbb{C}^k$ (resp. $\mathbb{R}^k$).

Remark. We could consider other basis, like differentiable manifolds, analytic spaces, infinite-dimensional mfds...

## Rigidity & cohomology.

Well-known facts:

1) $H^2(\mathcal{L},\mathcal{L})$ classifies infinitesimal deformations of $\mathcal{L}$.
2) If $\dim H^2(\mathcal{L},\mathcal{L}) < \infty$, then all formal deformations of $\mathcal{L}$ up to equivalence can be realized in this vectorspace.
3) If $H^2(\mathcal{L},\mathcal{L}) = 0$ then $\mathcal{L}$ is infinitesimally & formally rigid.
4) If $\dim \mathcal{L} < \infty$, then $H^2(\mathcal{L},\mathcal{L}) = 0 \Rightarrow \mathcal{L}$ is also rigid in geometric & analytic sense.

As our examples show, without the condition $\dim \mathcal{L} < \infty$, 4) is not true anymore.

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## Krichever–Novikov algebra

Witt–Virasoro, current-affine Lie algebras

Witt: consider of those meromorphic vector fields on the Riemann sphere which are holomorphic outside $\{0,\infty\}$. Basis & structure:

$$l_n = z^{n+1}\frac{d}{dz}, \quad n\in\mathbb{Z}, \qquad [l_n,l_m]=(m-n)l_{n+m}.$$

Virasoro: its universal central extension with the central element $c$:

$$[l_n,l_m]=(m-n)l_{n+m}+\frac{1}{12}(n^3-n)\delta_{n,-m}\,c, \quad [l_n,c]=0$$

Known: The Witt and Virasoro algebra are formally rigid. (F.)

Current algebra: $\bar{g}=g\otimes\mathbb{C}[z^{-1},z]$, where $g$ is a finite dimensional Lie algebra, let it be simple.

Affine Lie algebra: with the Cartan–Killing form $\beta$ the central extension $\hat{g}$ with

$$[x\otimes z^n, y\otimes z^m]=[x,y]\otimes z^{n+m}-\beta(x,y)\cdot n\cdot\delta_m^{-n}\cdot c,$$
$$[c,\hat{g}]=0$$

Known: For $g$ finite-dimensional simple, $\hat{g}$ remain formally rigid. (Lecomte, Roger)

EX. Let $M$ be a compact Riemann surface of genus $g$, or, a smooth projective curve over $\mathbb{C}$. Let $N, K\in\mathbb{N}$, $N\geq 2$ & $1\leq K<N$ numbers. Fix

$$I=(P_1,\ldots,P_K), \quad O=(Q_1,\ldots,Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points") on the curve. Assume $P_i\neq Q_j$ $\forall(i,j)$. $I$ = in-points, $O$ = out-points. We can consider $I, O$ as a set & $A=I\cup O$ also as a set.

Let $\mathcal{A}$ be the associative algebra consisting of those meromorphic functions on $M$ which are holomorphic outside the set of points $A$ with pointwise multiplication.

Let $\mathcal{L}$ be the Lie algebra consisting of those meromorphic vector fields which are holomorphic outside of $A$ with the usual Lie bracket of vector fields. VF algebra of Krichever-Novikov type

Remark. They are both infinite dimensional algebras.

Higher genus, multi-point current algebra of Krichever-Novikov type

Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra. Consider the tensor product $\overline{\mathcal{G}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \quad x, y \in \mathfrak{g}, \ f, g \in \mathcal{A}.$$

The algebra $\overline{\mathcal{G}}$ is the higher genus current algebra. It is infinite-dimensional, & might be considered as the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on the Riemann surface with only poles outside of $A$.

The classical genus $0$ & $N=2$ point case is given by the geometric data:

$$M = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{z=0\}, \quad O = \{z=\infty\}$$

In this case the algebras are the well-known ones. For the function algebra we get $\mathcal{A} = \mathbb{C}[z^{-1}, z]$. The vector field algebra $\mathcal{L}$ is the Witt algebra $W$. The current algebra $\overline{\mathcal{G}}$ is the standard current algebra $\overline{\mathfrak{g}}$.

In the classical situation the algebras are obviously graded, $\deg l_n := n$, $\deg x \otimes z^n := n$. For higher genus there is usually no grading. But there exist a weaker concept which works.

Def. Let $\mathcal{A}$ be an (associative or Lie) algebra admitting a direct decomposition as vector space: $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$. The algebra $\mathcal{A}$ is called almost graded if

1) $\dim \mathcal{A}_n < \infty$
2) $\exists\ R, S$ constants s.t. $\mathcal{A}_n \cdot \mathcal{A}_m \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{A}_h \quad \forall n,m \in \mathbb{Z}$

The elements of $\mathcal{A}_n$ are called homogeneous elements of degree $n$.

Prop. The algebras $\mathcal{L}, \mathcal{A}, \overline{\mathcal{G}}$ are almost graded. The almost-grading depends on the splitting $A = I \cup O$. (Schl.)

Some geometric deformations

1) Complex torus $T$

Consider the genus 1 case – elliptic curve case.

Let $\tau \in \mathbb{C}$ with $\operatorname{im}\tau > 0$ &

$$L = \langle 1, \tau \rangle_{\mathbb{Z}} := \{ m + n\cdot\tau \mid m,n \in \mathbb{Z} \} \subset \mathbb{C} \text{ lattice.}$$

Then $T = \mathbb{C}/L$. It carries a natural structure of a complex manifold coming from the structure of $\mathbb{C}$. It will be a compact Riemann surface of genus 1.

The field of meromorphic functions on $T$ is generated by the Weierstrass $\wp$ function. Let

$$\wp\left(\tfrac{1}{2}\right) = e_1, \quad \wp\left(\tfrac{\tau}{2}\right) = e_2, \quad \wp\left(\tfrac{\tau+1}{2}\right) = e_3 .$$

$$(e_1 + e_2 + e_3 = 0)$$

Then $\wp'$ fulfills the differential equation

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) .$$

The function $\wp$ is an even meromorphic function with poles of order two at the points of the lattice and holomorphic elsewhere.

$\wp'$ is an odd meromorphic function with poles of order 3 at the points of the lattice & holomorphic elsewhere

The map

$$T \to \mathbb{P}^2(\mathbb{C}), \quad z \bmod L \mapsto \begin{cases} (\wp(z) : \wp'(z) : 1) & z \notin L \\ (0:1:0) & z \in L \end{cases}$$

realizes $T$ as a complex-algebraic smooth curve in the projective plane. As its genus is one it is an elliptic curve. From the differential equation for the derivative it follows that the affine part of the curve can be given by the smooth cubic curve

$$Y^2 = 4(X-e_1)(X-e_2)(X-e_3) =: f(X).$$

The point at infinity on the curve is the point $\infty = (0:1:0)$. For our purpose it is enough to consider two marked points. We will always put one marked point to $\infty$, the other to the point with the affine coordinate $(e_1, 0)$.

2) Vector field algebra

Consider the vector field algebra $\mathcal{L}$. A basis is given by

$$V_{2k+1} := (X-e_1)^k Y \frac{d}{dX}, \quad V_{2k} := \frac{1}{2} f(X)(X-e_1)^{k-2} \frac{d}{dX}$$

$$k \in \mathbb{Z}.$$

If we vary the points $e_1$ & $e_2$, we obtain families of curves & associated families of vector field algebras (as long as the curves are nonsingular).

Consider the straight lines

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \; s \in \mathbb{C}, \quad D_\infty := \{(0, e_2) \in \mathbb{C}^2\},$$

& the open set

$$B = \mathbb{C}^2 \setminus (D_1 \cup D_{-\frac{1}{2}} \cup D_{-2}) \subset \mathbb{C}^2.$$

The curves are nonsingular exactly over the points of $B$.

For the vector field algebra we get:

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & n, m \text{ odd} \\ (m-n)\left(V_{n+m} + 3e_1 V_{n+m-2} + (e_1-e_2)(e_1-e_3)V_{n+m-4}\right) & n, m \text{ even} \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} + (m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4} & n \text{ odd}, m \text{ even.} \end{cases}$$

Denote by $\underline{\mathcal{L}^{(e_1, e_2)}}$ the Lie algebra for every pair $(e_1, e_2) \in \mathbb{C}^2$. Obviously, $\mathcal{L}^{(0,0)} \cong W$.

__Prop__. for $(e_1, e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra, but $\mathcal{L}^{(0,0)} \cong W$.

If we restrict our 2-dimensional family to a line $D_s$ ($s \neq \infty$), then we obtain a one-dimensional family

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & n, m \text{ odd} \\ (m-n)\left(V_{n+m} + 3e_1 V_{n+m-2} + e_1^2(1-s)(2+s)V_{n+m-4}\right) & n, m \text{ even} \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} + (m-n-2)e_1^2(1-s)(2+s)V_{n+m-4} & n \text{ odd}, m \text{ even.} \end{cases}$$

Here $s$ has a fixed value & $e_1$ is the deformation parameter. It can be shown that if $e_1 \neq 0$, the algebras over two points in $D_s$ are pairwise isomorphic but not isomorphic to the algebra over 0.

$\Rightarrow$ __Theorem.__ $W$ admits deformations $\mathcal{L}_t$ over the affine line with $\mathcal{L}_0 \cong W$, which, restricted to every neighborhood of $t=0$, are non-trivial.

# The current algebra.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra & $\mathcal{A}$ the algebra of meromorphic functions corresponding to the geometric situation we introduced.
A basis for $\mathcal{A}$ is given by

$$A_{2k} = (X - e_1)^k, \qquad A_{2k+1} = \frac{1}{2} Y \cdot (X - e_1)^{k-1}, \quad k \in \mathbb{Z}.$$

The bracket for the elements of $\overline{\mathcal{G}}$:

$$[x \otimes A_n, y \otimes A_m] = \begin{cases} [x,y] \otimes A_{n+m} & n \text{ or } m \text{ even} \\ [x,y] \otimes A_{n+m} + 3e_1 [x,y] \otimes A_{n+m-2} \\ \quad + (e_1 - e_2)(2e_1 + e_2)[x,y] \otimes A_{n+m-4} & n, m \text{ odd}. \end{cases}$$

If we let $e_1$ & $e_2$ (and hence $e_3$) go to zero, we obtain the classical current algebra as degeneration.

**Theorem.** The family, even if restricted on $D_s$, is locally non-trivial. So the current algebra is neither geometrically, nor analytically rigid.

$M$ – compact Riemann surface of genus $g$
$\mathcal{A}$ – assoc. algebra consisting of those meromorphic functions on $M$ which are holomorphic outside the set of points $A$.

## Open questions

1. More general global base ?
   Infinite-dimensional base ?
2. Families obtained by geometric processes describe much better the local neighborhood of the moduli space of a given infinitesimal Lie algebra.
   Other geometric processes ?.
3. What is the right deformation theory ?
   May be Kontsevich's concept of semi-formal deformations (related to filtrations of certain type ?
   (Our almost grading of the families are semi-formal deformations in his sense.)