Algebraic aspects in geometry;

Będlewo, October 2007

# Dirac operators on noncommutative manifolds

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Commutativity of algebra of functions on space $X$
is
localization of points of X

Gel'fand-Naimark:
equivalence of categories $C_0(X) \leftrightarrow X$,

commutative $C^*$-algebras $\leftrightarrow$ loc. comp. Hausdorff spaces;
points $x \in X$ are characters (or 1-dim irreps) of $X$

$$x(f) := f(x)$$

Noncommutativity

$$[p, x] = 1 \qquad \Rightarrow \qquad \Delta p \Delta x \geq 1$$

localization of points is ruled out

Pathological spaces:

Equivalence relation $\mathcal{R}$ on $X$;
the quotient $Y = X/\mathcal{R}$ can be bad even for good $X$

Classically: function on the quotient

$$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \quad ; \quad f \quad \text{is} \quad \mathcal{R} - \text{invariant}\}$$

often not many, only constant functions: $\mathcal{A}(Y) = \mathbb{C}$

NCG: a noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

of functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation (compact support, of rapid decay,...)

convolution product:

$$(f_1 * f_2)(x,y) = \sum_{x\sim u\sim y} f_1(x,u) f_2(u,y)$$

involution:

$$f^*(x,y) = \overline{f(y,x)}$$

the quotient $Y = X/\mathcal{R}$ is a noncommutative space with a noncommutative algebra of functions $\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$;

as good as $X$ to do geometry:
exterior forms, metric, integration, vector bundles, connections, curvature, ...

with new phenomena coming from noncommutativity

The celebrated noncommutative torus aka
the irrational rotation algebra

$$\mathcal{A}_\theta = C^\infty(\mathbb{T}^2_\theta) \simeq C^\infty(S^1/\theta\mathbb{Z})$$

More general examples

Toric noncommutative manifolds a.k.a. isospectral deformations

deformations of a classical Riemannian mfld;
satisfy all properties of a nc spin geometry:

**Theorem 1 (Connes-L.).** *Let $M$ be a compact Riemannian spin manifold (no boundary) whose isometry group has rank $r \geq 2$. Then $M$ admits natural isospectral deformations to noncommutative geometries $M_\theta$.*

$$\theta = (\theta_{jk} = -\theta_{kj}) \qquad \theta_{jk} \in \mathbb{R} \qquad j,k = 1,\dots,r$$

idea: deform the standard spectral triple describing the Riemannian geometry of $M$ along a torus embedded in the isometry group to get an isospectral triple

$$(C^\infty(M_\theta), \mathcal{H}, D, \gamma)$$

recent noncompact examples

Deforming a torus action

$M$ an $m$ dim compact mfld no boundary (Riemannian, spin)

an isometric smooth action $\sigma$ of $\mathbb{T}^r$, $n \geq 2$

decompose $C^\infty(M)$ into spectral subspaces indexed by the dual group $\mathbb{Z}^r = \widetilde{\mathbb{T}}^r$: a $t \in \mathbb{Z}^r$ labels a character of $\mathbb{T}^r$

$$e^{2\pi i s} \mapsto e^{2\pi i\, t \cdot s}$$

the $r$-th spectral subspace for $\sigma$ on $C^\infty(M)$: functions $f_t$ s.t.

$$\sigma_s(f_t) = e^{2\pi \mathrm{i}\, t \cdot s}\, f_t,$$

for each $f \in C^\infty(M)$, a sum $f = \sum_{t \in \mathbb{Z}^r} f_t$;
a unique rapidly convergent series

$\theta = (\theta_{jk} = -\theta_{kj})$ a real antisymmetric $r \times r$ matrix

the $\theta$-deformation of $C^\infty(M)$: replace the ordinary product by a deformed product. on spectral subspaces is given by

$$f_t \times_\theta g_{t'} := f_t \, \sigma_{\frac{1}{2}t\cdot\theta}(g_{t'}) = e^{\pi \mathrm{i} \, t\cdot\theta\cdot t'} f_t g_{t'},$$

denote $$C^\infty(M_\theta) := (C^\infty(M), \times_\theta)$$

the action $\sigma$ of $\mathbb{T}^r$ extends to $C^\infty(M_\theta)$

at the level of the $C^*$-algebra of continuous functions one has a strict deformation quantization in the direction of the Poisson structure defined by the matrix $\theta$ (cf. Rieffel) the action of $\mathbb{T}^r$ on $C^\infty(M)$ extends to an action on the deRham complex $\Omega^*(M)$ commuting with the exterion derivative $d$

with the same techniques, one deforms the exterior product, while unchanging $d$ to get a complex

$(\Omega^*(M_\theta), d)$

with $\Omega^0(M_\theta) = C^\infty(M_\theta)$

it is not graded commutative in general

The Hodge operator $*$ on $\Omega^*(M)$ of the Riemannian metric is twisted to an Hodge operator $*_\theta$ on $\Omega^*(M_\theta)$

## Finite summable spectral triple
## (noncommutative spin geometries)

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is:

- a cmpl unital $*$-algebra $\mathcal{A}$ with a faithful $*$-rep. $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a (separable) Hilbert space,

- a self-adjoint operator $D$ on $\mathcal{H}$ ("the Dirac operator") s. t.

(i) $(D + \mathrm{i})^{-1}$ is compact

(ii) $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$

The spectral triple is *graded* (or even) if there exists a $\mathbb{Z}_2$-grading operator $\gamma$ on $\mathcal{H}$, $\gamma = \gamma^*$, $\gamma^2 = 1$, s.t.

$$\gamma D = -D\gamma, \qquad \pi(a)\gamma = \gamma\pi(a), \qquad a \in \mathcal{A}$$

With $0 < \mu < \infty$, the spectral triple is $\mu^+$-summable (of metric dimension $\mu$) if $(D^2 + 1)^{-1/2}$ is in the Dixmier ideal $\mathcal{L}^{\mu+}(\mathcal{H})$.

$M$ a compact Riemannian spin manifold (no boundary)

the canonical spectral triple on $M$

$\mathcal{H} := L^2(M, \mathcal{S})$ the Hilbert space of spinors;

$D$ the Dirac operator of the metric of $M$;

$C^\infty(M)$ act on spinors pointwisely:

$$\Rightarrow \qquad (C^\infty(M), \mathcal{H}, D)$$

Back to toric nc manifolds

From the canonical spectral triple on $M$ : $\quad (C^\infty(M), \mathcal{H}, D)$

a double cover $c : \widetilde{\mathbb{T}}^r \to \mathbb{T}^r$ and a representation of $\tilde{\mathbb{T}}^r$ on $\mathcal{H}$ by unitary operators $U(s)$, s.t

$$U(s) D U(s)^{-1} = D, \qquad U(s)\pi(f)U(s)^{-1} = \pi(\sigma_{c(s)}(f)).$$

$P = (p_1, p_2, \ldots, p_r)$ infinitesimal gen.s of the toric action

$$U(s) = \exp\left(2\pi \mathrm{i}\ s \cdot P\right)$$

For $T \in \mathcal{B}(\mathcal{H})$ an action $\tilde{\mathbb{T}}^r \ni s \mapsto \alpha_s(T) := U(s)TU(s)^{-1}$

a spectral decomposition of $T \in \mathcal{B}(\mathcal{H})$: $T = \sum T_t$ ; $t \in \mathbb{Z}^r$ and $T_t$ is homogeneous of degree $r$ for the action of $\tilde{\mathbb{T}}^r$:

$$\alpha_s(T_t) = e^{2\pi \mathrm{i} r \cdot s} T_t, \quad \forall \quad s \in \tilde{\mathbb{T}}^r$$

a twisted representation on $\mathcal{H}$ of the smooth elements of $\mathcal{B}(\mathcal{H})$

$$L_\theta(T) := \sum\nolimits_t T_t \exp\left\{\pi \mathrm{i}\, t_j \theta_{jk} p_k\right\}$$

in particular we have $L_\theta(C^\infty(M))$;
it is isomorphic, as an algebra, to $C^\infty(M_\theta)$:

$$L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g)$$

think of $L_\theta$ as a *quantization map* from

$$L_\theta : C^\infty(M) \to C^\infty(M_\theta)$$

the datum $(L_\theta(C^\infty(M)), \mathcal{H}, D)$ is a nc spin geometry
(also a twisted real structure $J$
and a $\mathbb{Z}_2$-grading $\gamma$ of $\mathcal{H}$ – for the even case)

all spectral properties are unchanged;

the triples are $m^+$-summable (of metric dimension $m$)
$m = \dim M$

## Some recent experimental findings: equivariant spectral triples

The guiding principle:

equivariance with respect to a 'quantum symmetry'
this will build all the geometry from scratch

## Symmetries

implemented by the action of a Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{g})$, a quantum universal enveloping algebra;

Let $\mathcal{U} = (\mathcal{U}, \Delta, S, \varepsilon)$ be a Hopf $*$-algebra and $\mathcal{A}$ be a left $\mathcal{U}$-module $*$-algebra, i.e., there is a left action $\triangleright$ of $\mathcal{U}$ on $\mathcal{A}$,

$$h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y),$$

$$h \triangleright 1 = \varepsilon(h)1, \quad (h \triangleright x)^* = S(h)^* \triangleright x^*,$$

notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

A $\mathcal{U}$-equivariant $*$-representation $\pi$ of $\mathcal{A}$ on $\mathcal{V}$ (a dense subspace of $\mathcal{H}$) is a $*$-representation of the left crossed product $*$-algebra

$$\mathcal{A} \rtimes \mathcal{U}$$

defined as the $*$-algebra generated by the two $*$-subalgebras $\mathcal{A}$ and $\mathcal{U}$ with crossed commutation relations

$$hx = (h_{(1)} \triangleright x)h_{(2)}, \quad h \in \mathcal{U},\ x \in \mathcal{A}$$

that is, there is also a $*$-representation $\lambda$ of $\mathcal{U}$ on $\mathcal{V}$ s. t.

$$\lambda(h)\,\pi(x)\,\xi = \pi(h_{(1)} \triangleright x)\,\lambda(h_{(2)})\,\xi,$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in \mathcal{V}$.

A linear operator $D$ on $\mathcal{V}$ is equivariant if it commutes with $\lambda(h)$, for all $h \in \mathcal{U}$ and $\xi \in \mathcal{V}$

$$D\lambda(h)\,\xi = \lambda(h)D\,\xi$$

Examples:

toric noncommutative manifolds
(including and generalizing nc tori)

the manifold of quantum SU(2)

families of quantum two spheres

higher dimensional quantum orthogonal spheres

quantum projective spaces

A recent general strategy for
isospectral Dirac operators on compact quantum groups
(deforming simply connected simple compact Lie groups)

via a Drinfel'd twist

Neshveyev-Tuset, OA/0703161 ;

the quantum Dirac operator is of the form

$$D_q = \mathcal{F}\, D\, \mathcal{F}^{-1} , \qquad \Rightarrow \ \text{Spec}(D_q) = \text{Spec}(D)$$

with $\mathcal{F}$ a Drinfel'd twist implementing the deformed coproduct and antipode on the symmetry Hopf algebra $\mathcal{U}_q(LieG))$

“ There exists formulas for $q$-analogues of the Dirac operator on quantum groups ..... ; let us call $Q$ these “naive” Dirac operators. Now the fundamental equation to define the thought for true Dirac operator $D$ which we used above implicitly on the deformed 3-sphere (after suspension to the 4-sphere and for deformation parameters which are complex of modulus one) is,

$$[D]_{q^2} = Q\,.$$

where the symbol $[x]_q$ has the usual meaning in $q$-analogues, ... The main point is that it is only by virtue of this equation that the commutators $[D, a]$ will be bounded .... ”

A. Connes, G. L., Noncommutative Manifolds, the Instanton Algebra and Isospectral Deformations, CMP (2001).

After some initial skepticism, this programme was completed in

L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J.C. Varilly
The Dirac operator on $SU_q(2)$, CMP (2005).

An isospectral nc geometry on the 'manifold underlying' $SU_q(2)$

**Proposition 2.** *The spectral triple* $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ *is regular and* $3^+$*-summable. It has simple dimension spectrum given by* $\Sigma = \{1, 2, 3\}$.
*Its KO-dimension is 3.*

The equivariance is with respect to an action of the quantum universal envelopping algebra $\mathcal{U} = \mathcal{U}_q(\mathrm{su}(2))$

## The nc geometry of $\mathcal{A}(\mathrm{SU}_q(2))$

The algebra:

With $0 < q < 1$, let $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ be the $*$-algebra generated by $a$ and $b$, with relations:

$$ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b,$$

$$a^*a + q^2 b^*b = 1, \qquad aa^* + bb^* = 1$$

these state that the defining matrix is unitary

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$$

$\mathcal{A}(\mathrm{SU}_q(2))$ is a Hopf $*$-algebra (a quantum group) with

- coproduct:

$$\triangle \begin{pmatrix} a & b \\ -\,qb^* & a^* \end{pmatrix} := \begin{pmatrix} a & b \\ -\,qb^* & a^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ -\,qb^* & a^* \end{pmatrix}$$

- counit:

$$\varepsilon \begin{pmatrix} a & b \\ -\,qb^* & a^* \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- antipode:

$$S \begin{pmatrix} a & b \\ -\,qb^* & a^* \end{pmatrix} = \begin{pmatrix} a^* & -\,qb \\ b^* & a \end{pmatrix}$$

The symmetry:

The quantum universal envelopping algebra $\mathcal{U} = \mathcal{U}_q(\mathrm{su}(2))$ is the $*$-algebra generated by $e, f, k$, with $k$ invertible, and relations

$$ek = qke, \qquad kf = qfk,$$

$$k^2 - k^{-2} = (q - q^{-1})(fe - ef),$$

the $*$-structure is simply

$$k^* = k, \quad f^* = e, \quad e^* = f$$

The Hopf $*$-algebra structure

- coproduct:

$$\Delta k = k \otimes k, \quad \Delta f = f \otimes k + k^{-1} \otimes f, \quad \Delta e = e \otimes k + k^{-1} \otimes e$$

- counit:

$$\epsilon(k) = 1, \quad \epsilon(f) = 0, \quad \epsilon(e) = 0$$

- antipode:

$$Sk = k^{-1}, \quad Sf = -qf, \quad Se = -q^{-1}e$$

## The action of $\mathcal{U}$ on $\mathcal{A}$

A natural bilinear pairing between $\mathcal{U}$ and $\mathcal{A}$,

$$\langle k, a\rangle = q^{\frac{1}{2}}, \quad \langle k, a^*\rangle = q^{-\frac{1}{2}}, \qquad \langle e, -qb^*\rangle = \langle f, b\rangle = 1$$

gives commuting l. and r. $\mathcal{U}$-module algebra structures on $\mathcal{A}$:

$$h \triangleright x := x_{(1)} \langle h, x_{(2)}\rangle, \qquad x \triangleleft h := \langle h, x_{(1)}\rangle\, x_{(2)}$$

we use the notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$

The invertible antipode transforms the right action $\triangleleft$ into a second left action of $\mathcal{U}$ on $\mathcal{A}$, commuting with the first

$$h \cdot x := x \triangleleft S^{-1}(\vartheta(h)),$$

with the autom. $\vartheta$, $\qquad \vartheta(k) := k^{-1}, \quad \vartheta(f) := -e, \quad \vartheta(e) := -f$

## The representation theory of $\mathcal{U}_q(\mathrm{su}(2))$

The irreducible finite dim representations $\sigma_l$ of $\mathcal{U}_q(\mathrm{su}(2))$ are labelled by nonnegative half-integers (the spin) $l \in \frac{1}{2}\mathbb{N}_0$:

$$\begin{aligned}
\sigma_l(k)\,|lm\rangle &= q^m\,|lm\rangle, \\
\sigma_l(f)\,|lm\rangle &= \sqrt{[l-m][l+m+1]}\,|l,m+1\rangle, \\
\sigma_l(e)\,|lm\rangle &= \sqrt{[l-m+1][l+m]}\,|l,m-1\rangle,
\end{aligned}$$

on the irreducible $\mathcal{U}$-module $\quad \mathcal{V}_l = \mathrm{span}\{|lm\rangle,\, m=-l,\dots,l\}$

$\sigma_l$ is a $*$-representation, with respect to the hermitian scalar product on $\mathcal{V}_l$ for which the vectors $|lm\rangle$ are orthonormal.

the brackets denote $q$-integers

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{provided} \quad q \neq 1$$

## Left regular representation of $\mathcal{A}(\mathrm{SU}_q(2))$

The left regular representation of $\mathcal{A}$ as an equivariant representation with respect to the two left actions $\triangleright$ and $\cdot$ of $\mathcal{U}$

With $\lambda$ and $\rho$ mutually commuting representations of $\mathcal{U}$ on $\mathcal{V}$, a representation $\pi$ of the $*$-algebra $\mathcal{A}$ on $\mathcal{V}$ is $(\lambda,\rho)$-*equivariant* if

$$\begin{aligned} &\lambda(h)\,\pi(x)\xi = \pi(h_{(1)}\cdot x)\,\lambda(h_{(2)})\xi, \\ &\rho(h)\,\pi(x)\xi = \pi(h_{(1)}\triangleright x)\,\rho(h_{(2)})\xi, \qquad \forall\ h\in\mathcal{U},\ x\in\mathcal{A},\ \xi\in\mathcal{V} \end{aligned}$$

As a representation space the preHilbert space

$$\mathcal{V} := \bigoplus_{2l=0}^{\infty} \mathcal{V}_l\otimes\mathcal{V}_l, \qquad |lmn\rangle := |lm\rangle\otimes|ln\rangle, \qquad m,n=-l,\dots,l$$

Two copies of $\mathcal{U}_q(\mathrm{su}(2))$ act via the irreps $\sigma_l$:

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h)$$

A $(\lambda, \rho)$-equivariant $*$-repr $\pi$ of $\mathcal{A}(\mathrm{SU}_q(2))$ on the Hilbert space $\mathcal{V}$ is the left regular representation. It has the form:

$$\begin{aligned}\pi(a)\,|lmn\rangle &= A^{+}_{lmn}|l^{+}m^{+}n^{+}\rangle + A^{-}_{lmn}|l^{-}m^{+}n^{+}\rangle,\\ \pi(b)\,|lmn\rangle &= B^{+}_{lmn}|l^{+}m^{+}n^{-}\rangle + B^{-}_{lmn}|l^{-}m^{+}n^{-}\rangle,\end{aligned}$$

with suitable explicit constants $A^{\pm}_{lmn}, B^{\pm}_{lmn}$.

Here $\qquad l^{\pm} := l \pm \frac{1}{2}, \quad m^{\pm} := m \pm \frac{1}{2}, \quad n^{\pm} := n \pm \frac{1}{2}$

## Spin representation

We amplify the left regular representation $\pi$ of $\mathcal{A}$ to the spinor representation $\pi' = \pi \otimes \mathrm{id}$ on

$$\mathcal{W} := \mathcal{V} \otimes \mathbb{C}^2 = \mathcal{V} \otimes \mathcal{V}_{\frac{1}{2}}$$

We also set $\rho' = \rho \otimes \mathrm{id}$ on $\mathcal{W}$.

But, we replace $\lambda$ on $\mathcal{V}$ by its tensor product with $\sigma_{\frac{1}{2}}$ on $\mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)})$$

The spinor representation $\pi'$ of $\mathcal{A}$ on $\mathcal{W}$ is $(\lambda', \rho')$-equivariant.

A basis $\{|j\mu n{\uparrow}\rangle, |j\mu n{\downarrow}\rangle\}$ for $\mathcal{W}$ from its $q$-Clebsch-Gordan decomp.

## Equivariant Dirac operator

Any self-adjoint operator on $\mathcal{H} = (\mathcal{V} \otimes C^2)^{cl}$, that commutes with both actions $\rho'$ and $\lambda'$ of $\mathcal{U}_q$(su(2)) is of the form

$$D|j\mu n{\uparrow}\rangle = d_j^{\uparrow}\,|j\mu n{\uparrow}\rangle, \qquad D|j\mu n{\downarrow}\rangle = d_j^{\downarrow}\,|j\mu n{\downarrow}\rangle,$$

with $d_j^{\uparrow}$ and $d_j^{\downarrow}$ real eigenvalues of $D$; depend only on $j$;

Restrictions on eigenvalues comes by requiring boundedness of the commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$

Unbounded commutators are obtained with the "naive $q$-Dirac"

$$d_j^{\uparrow} = \frac{2\,[2j+1]}{q+q^{-1}}, \qquad d_j^{\downarrow} = -d_j^{\uparrow} \qquad \text{[BibikovKulish]}$$

$q$-analogues of the classical eigenvalues of $\not{D} - \frac{1}{2}$;

$\not{D}$ is the classical ('round') Dirac operator on the sphere $\mathbb{S}^3$;

An operator $D$ with spectrum the one of the classical $\not{D}$ [CL]

With eigenvalues "linear in $j$":

$$d_j^{\uparrow} = c_1^{\uparrow} j + c_2^{\uparrow}, \qquad d_j^{\downarrow} = c_1^{\downarrow} j + c_2^{\downarrow},$$

with $c_1^{\downarrow} c_1^{\uparrow} < 0$ (for a nontrivial sign), all commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$ are bounded operators.

up to irrelevant scaling factors the choice of $c_j^{\uparrow}, c_j^{\downarrow}$ is immaterial; with $d_j^{\uparrow} = 2j + \frac{3}{2}, \qquad d_j^{\downarrow} = -2j - \frac{1}{2},$

the spectrum of $D$ (with multiplicity) is the classical one

Essentially the only possibility for a Dirac operator satisfying a (modified) first-order condition

A full analysis of the Local Index Formula in

L. Dabrowski, et al.,
The local index formula for $SU_q(2)$, K-Theory (2005).

The dimension spectrum is worked out via a symbol map from order zero pseudodifferential operators on the algebra describing the space to a noncommutative version of the cosphere bundle

For all cases:
an interesting pseudo-differential calculus

Formulæ for Connes-Moscovici local index thm

A lot of experimental evidence for a general strategy for Local Index Formulas on quantum groups and their homogeneous spaces

## The next level

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular if

$$\mathcal{A} \cup [D, \mathcal{A}] \quad \subset \quad \mathrm{OP}^0 := \bigcap_{j \in \mathbb{N}} \mathrm{dom}\, \delta^j \ ;$$

$\delta$ is the unbounded derivation on $\mathcal{B}(\mathcal{H})$ : $\delta(a) = [|D|, a]$.

For a regular spectral triple, pseudodifferential operators of order less or equal than zero make up the algebra $\Psi^0(\mathcal{A})$ generated by

$$\bigcup_{k \in \mathbb{N}} \delta^k(\mathcal{A} \cup [D, \mathcal{A}]);$$

it is a subalgebra of $\mathrm{OP}^0$, $\Psi^0(\mathcal{A}) \subset \mathrm{OP}^0$.

The "smoothing operators" $\mathrm{OP}^{-\infty}$ are a two-sided ideal in $\Psi^0(\mathcal{A})$,

$$\mathrm{OP}^{-\infty} := \{T \in \mathrm{OP}^0 : |D|^n T \in \mathrm{OP}^0 \ , \forall\, n \in \mathbb{N}\}$$

contribute holomorphic terms; can be dropped in from many computations. (in index computations)

Locality is insensitivity to smoothing perturbations:
a functional $\Phi : \Psi^0(\mathcal{A}) \to \mathbb{C}$ is local if

$$\Psi|_{\mathrm{OP}^{-\infty}} = 0$$

locality makes complicated expressions computable;
allows to neglect irrelevant details

For a spectral triple is of dimension $\mu$, the "zeta-type" functions

$$\zeta_a(z) := \mathrm{Tr}_{\mathcal{H}}\Big(a(|D|)^{-z}\Big), \qquad a \in \Psi^0(\mathcal{A}),$$

are holomorphic for $z \in \mathbb{C}$ with $\mathrm{Re}\, z > \mu$

The spectral triple has dimension spectrum $\Sigma \subset \mathbb{C}$, $\Sigma$ a countable set, if all $\zeta_a(z)$, $a \in \Psi^0(\mathcal{A})$, extend to meromorphic functions on $\mathbb{C}$ with poles in $\Sigma$ as unique singularities.

For $\Sigma$ made only of simple poles, a trace on $\Psi^0(\mathcal{A})$

$$⨍ T := \mathrm{Res}_{s=0}\, \mathrm{Tr}_{\mathcal{H}}\Big(T|D|^{-s}\Big)$$

skip this at a first reading

A real structure on $(\mathcal{A}, \mathcal{H}, D)$

We need to weaken the original definition in order to obtain a nontrivial spin geometry on the coordinate algebra of quantum groups and of associated quantum (homogeneous) spaces

A *real structure* $J$ for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an antilinear isometry $J$ on $\mathcal{H}$ such that

$$J^2 = \pm 1, \qquad JD = \pm DJ,$$

$$[\pi(a), J\pi(b)J^{-1}] \in \mathcal{I}, \qquad [[D, \pi(a)], J\pi(b)J^{-1}] \in \mathcal{I}, \qquad a, b \in \mathcal{A},$$

with $\mathcal{I}$ an operator ideal of 'infinitesimals'

If the spectral triple has grading $\gamma$, there is in addition

$$J\gamma = \pm\gamma J$$

The three signs above depend on the KO-dimension of the triple (modulo 8)

The original definition corresponds to $\mathcal{I} = 0$

Examples coming from quantum groups require $\mathcal{I} = \mathrm{OP}^{-\infty}$

skip this at a first reading

An antiunitary operator $J$ is equivariant if it leaves $\mathcal{V}$ invariant and if it is the antiunitary part in the polar decomposition of an antilinear (closed) operator $T$ that implements the antilinear involutory automorphism $h \mapsto (Sh)^*$, $S$ the antipode of $\mathcal{U}$, i.e.

$$T\lambda(h)\,\xi = \lambda(S(h)^*)T\,\xi,$$

for all $h \in \mathcal{U}$ and $\xi \in \mathcal{V}$.

A (real graded) spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ is $\mathcal{U}$-equivariant if the representation of $\mathcal{A}$ and the operators $D$ and $J$ are equivariant (and $\gamma$ commutes with the equivariant representation).

The local index formula for 3-dimensional geometries

A Fredholm index of the operator $D$, $\varphi : K_1(\mathcal{A}) \to \mathbb{Z}$.

$$\varphi([u]) := \mathrm{Index}(PuP) = \dim\ker PUP - \dim\ker PU^*P.$$

With $F = \mathrm{Sign}\, D$ and $P = \frac{1}{2}(1 + F)$.

Computed by pairing $K_1(\mathcal{A})$ with "a nonlocal" cyclic cocycle

$$\chi_1(a_0, a_1) = \mathrm{Tr}(a_0\, [F, a_1]) \; ;$$

the Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ is 1-summable since all commutators $[F, \pi(x)]$ are trace-class

The C–M local index theorem expresses the index map in terms of a local cocycle $\phi_{\mathrm{odd}} = (\phi_1, \phi_2)$

$$\phi_1(a_0, a_1) := \int\!\!\!\!\!\!- a_0\,[D, a_1]\,|D|^{-1} - \frac{1}{4}\int\!\!\!\!\!\!- a_0\,\nabla([D, a_1])\,|D|^{-3} + \frac{1}{8}\int\!\!\!\!\!\!- a_0\,\nabla^2([D, a_1])\,|D|^{-5},$$

$$\phi_3(a_0, a_1, a_2, a_3) := \frac{1}{12}\int\!\!\!\!\!\!- a_0\,[D, a_1]\,[D, a_2]\,[D, a_3]\,|D|^{-3},$$

$$\nabla(T) := [D^2, T]$$

With $[F, a]$ traceclass for each $a \in \mathcal{A}$,

$$\phi_1(a_0, a_1) = \int\!\!\!\!\!\!- a_0\,\delta(a_1)F|D|^{-1} - \frac{1}{2}\int\!\!\!\!\!\!- a_0\,\delta^2(a_1)F|D|^{-2} + \frac{1}{4}\int\!\!\!\!\!\!- a_0\,\delta^3(a_1)F|D|^{-3},$$

$$\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12}\int\!\!\!\!\!\!- a_0\,\delta(a_1)\,\delta(a_2)\,\delta(a_3)F|D|^{-3}.$$

With the additional use of a simple dimension spectrum not containing 0 and bounded above by 3 ; the Chern character $\chi_1$ is equal to $\phi_{\rm odd} - (b+B)\phi_{\rm ev}$ where the $\eta$-cochain $\phi_{\rm ev} = (\phi_0, \phi_2)$ is

$$\phi_0(a) := \mathrm{Tr}(Fa\,|D|^{-z})\Big|_{z=0},$$

$$\phi_2(a_0, a_1, a_2) := \frac{1}{24} ⨍ a_0\,\delta(a_1)\,\delta^2(a_2)F|D|^{-3};$$

$$\phi_1 = \chi_1 + b\phi_0 + B\phi_2, \qquad \phi_3 = b\phi_2.$$

With the same conditions on the dimension spectrum and commutators $[F, a]$, the local Chern character $\phi_{\rm odd} = \psi_1 - (b+B)\phi'_{\rm ev}$,

$$\psi_1(a_0, a_1) := 2 ⨍ a_0\,\delta(a_1)P|D|^{-1} - ⨍ a_0\,\delta^2(a_1)P|D|^{-2}$$
$$+\frac{2}{3} ⨍ a_0\,\delta^3(a_1)P|D|^{-3},$$

and $\phi'_{\text{ev}} = (\phi'_0, \phi'_2)$,

$$\phi'_0(a) := \mathrm{Tr}(a\,|D|^{-z})\Big|_{z=0},$$
$$\phi'_2(a_0, a_1, a_2) := -\frac{1}{24} ⨍ a_0\,\delta(a_1)\,\delta^2(a_2) F|D|^{-3}.$$

The term in $P|D|^{-3}$ would vanish if the latter were traceclass [Connes] (this is the statement that $P$ has metric dimension 2)

Summing up, up to coboundaries, the cyclic 1-cocycles $\chi_1$ can be given by means of one single $(b, B)$-cocycle $\psi_1$:

$$\chi_1 = \psi_1 - b\beta, \quad \text{where} \quad \beta(a) = 2\,\mathrm{Tr}(Pa\,|D|^{-z})\Big|_{z=0}$$

a vast beautiful new territory out there

that cries to be explored

thank you for the moment !!