Będlewo. October 19, 2007

Glenn Barnich

Physique théorique et mathématique

Université Libre de Bruxelles &
International Solvay Institutes

# Algebraic structure of gauge systems: Theory and Applications

Why gauge theories ?

BRST differential for finite-dimensional toy model

Field theory: locality and examples

Anomalies, divergences, consistent deformations

Characteristic cohomology, central extensions in gravitational theories

So, thanks very much for the occasion to speak at this conference.

Today I would like to review a body of work that has been done since the mid-seventies on the structure of gauge theories. Gauge theories feature prominently in theoretical physics because the four known fundamental interactions, electromagnetism, the weak and strong nuclear forces, general relativity, and various unifying models such as string or higher spin theories, are described by theories of this type.

I will start by discussing the construction of the BRST differential in a finite-dimensional toy model. I will then move to the relevant case of field theory, show how to deal with locality and provide some standard examples of the construction.

The next part is devoted to standard applications of BRST cohomology in problems of physical interest, such as anomalies and divergences in quantum field theory, or the problem of consistent deformations in classical field theory. I will then discuss results on the so-called characteristic cohomology, and also, time permitting, more recent applications, in the context of asymptotic symmetries of gravitational theories.

## Theory I: Finite-dimensional toy model   Symetries & Stationary surface

function $S_0[\phi^i]$ on manifold $F$ "action"

vector fields $\mathcal{S} \ni \vec{v}, \quad \vec{v}S_0 = 0$ "symmetries"

stationary surface $\Sigma \subset F : \quad \frac{\partial S_0}{\partial \phi^i} = 0$ "shell"

symmetries induce well-defined vector fields $\mathcal{S}_\Sigma$ "on-shell symmetries"

regularity conditions imply that $\mathcal{S}_\Sigma \equiv \Gamma(T\Sigma)$

de Rham differential on $\Sigma : \quad \gamma$ "longitudinal differential"

NB: $\dim \mathrm{F} = \mathrm{N}$ $\quad rank \frac{\partial^2 S_0}{\partial \phi^i \phi^j} = N - M$ $\quad dim(\Sigma) = M$

aim of BRST construction: off-shell description of $H(\gamma)$.

Consider a function on a manifold, F, called action in the physics literature and the vector fields that leave this function invariant. They are called symmetries. The surface defined by the stationary points of the action is called Sigma and assumed to satisfy suitable regularity conditions. The symmetries induce well-defined vector fields on this surface.

Using the regularity conditions, one can show that the on-shell symmetries coincide with the vector fields tangent to Sigma.

The de Rham cohomology on \Sigma is denoted by \gamma and called the longitudinal differential.

Note that the whole construction really useful only if the Hessian is not of maximal rank, because otherwise \Sigma reduces to a point.

The aim of the BRST construction is to describe this differential without having to solve equations. In other words, one would like to construct a differential on a complex with coefficients that are functions on F whose cohomology reproduces the de Rham cohomology of the stationary surface \Sigma.

## Theory I: Finite-dimensional toy model Koszul-Tate resolution

symmetries $\vec{e}_\alpha = R^i_\alpha \frac{\partial}{\partial \phi^i}$ on $F$ generating $\Gamma(T\Sigma)$ "generating set"

all symmetries contain trivial ones $\mathcal{S} \ni \vec{v} \Longrightarrow \vec{v} = f^\alpha \vec{e}_\alpha + \mu^{[ij]} \frac{\partial S_0}{\partial \phi^j} \frac{\partial}{\partial \phi^i}$

on-shell closure of generating set $[\vec{e}_\alpha, \vec{e}_\beta] \approx f^\gamma_{\alpha\beta} \vec{e}_\gamma$

dual one-forms $C^\alpha$ "ghosts"

longitudinal differential $\gamma = C^\alpha \vec{e}_\alpha - \frac{1}{2} C^\alpha C^\beta f^\gamma_{\alpha\beta} \frac{\partial}{\partial C^\gamma}$, $\gamma^2 \approx 0$

additional generators $\phi^*_i, C^*_\alpha \leftrightarrow \frac{\partial}{\partial \phi^i}, \frac{\partial}{\partial C^\alpha}$ "antifields"

$$\delta = \frac{\partial S_0}{\partial \phi^i} \frac{\partial}{\partial \phi^*_i} + \phi^*_i R^i_\alpha \frac{\partial}{\partial C^*_\alpha} \qquad H\big(\delta, C^\infty(F) \otimes \wedge(\phi^*_i, C^*_\alpha)\big) \cong C^\infty(\Sigma)$$

"Koszul-Tate resolution"

For that, one needs a set of symmetries on F that, when restricted to \Sigma generate the vector fields on \Sigma over functions on \Sigma. Off-shell, all symmetries can be shown to be either a combination of this generating set or trivial, on-shell vanishing symmetries, and the generating set closes in general only up to such trivial symmetries. In these terms, the dual 1-forms are denoted by C^\alpha and called ghosts and de Rham differential on \Sigma takes the following form.

One then introduces additional generators called antifields. They are like the basic vector fields with shifted parity. The Koszul-Tate resolution \delta is then defined in such a way that its homology, defined in the space of graded polynomials in this generators with coefficients that are functions on F is isomorphic to functions on \Sigma in degree 0 and trivial in higher degree.

## Theory I: Finite-dimensional toy model BV complex

homological perturbation theory $s=\delta+\gamma+\dots,\qquad s^2=0$

$$H\big(s, C^\infty(F)\otimes\wedge(C^a,\phi^*_i,C^*_\alpha)\big)\cong H(\gamma, C^\infty(\Sigma)\otimes\wedge(C^a))$$ "BV complex"

Gerstenhaber algebra $(A,B)=\frac{\partial^R A}{\partial\phi^A}\frac{\partial^L B}{\partial\phi^*_A}-(\phi\leftrightarrow\phi^*)\qquad \phi^A\equiv(\phi^i,C^\alpha)$

"antibracket (degree 1)"

canonical generator (degree 0) $s=(S,\cdot),\qquad \frac{1}{2}(S,S)=0$

$$S=S_0+\phi^*_i R^i_\alpha C^\alpha+\frac{1}{2}C^*_\gamma f^\gamma_{\alpha\beta}C^\alpha C^\beta+\dots$$ "solution of classical master equation"

Henneaux & Teitelboim, Quantization of gauge systems

Using standard homological techniques, one then combines \delta with the off-shell version of \gamma, which was not a differential, into a differential on the space of graded polynomials in ghosts and antifields with coefficents that are functions on F whose cohomology reproduces the de Rham cohomology of the stationary surface.

In this Lagrangian case, there is additional structure, the space on which the BRST differential acts is a Gerstenhaber algebra for the so-called antibracket which has degree 1 and the BRST differential admits a canonical generator of degree 0, the solution of the classical master equation, which starts off like the classical action and then contains the generating set contracted with antifields and ghost. It is completed by higher order terms that depend on how complicated the commutator algebra of the generating set is off -shell.

## Theory I: Finite-dimensional toy model — Deformation theory

antibracket in cohomology $(\cdot,\cdot)_M : H^{g_1} \times H^{g_2} \longrightarrow H^{g_1+g_2+1}$

$$([A],[B])_M = [(A,B)]$$

deformation theory $S = \overset{(0)}{S} + \overset{(1)}{S} + \overset{(2)}{S} + \dots$ $\qquad \frac{1}{2}(\overset{(0)}{S},\overset{(0)}{S}) = 0$

non trivial infinitesimal deformations $[\overset{(1)}{S}] \in H^0(s)$

no obstruction if $\frac{1}{2}([\overset{(1)}{S}],[\overset{(1)}{S}])_M = [0] \in H^1(s)$

Because the differential is canonically generated, there is a well defined bracket in cohomology.

The usual deformation theory can of course be used in this context. When trying to deform an initial solution of the master equation, non trivial infinitesimal deformations are controlled by H0 and obstructions to continuing such deformations are controlled by the antibracket map induced in cohomology in degree 1

## Theory 2: Field theory Jet-bundles

classical mechanics: action functional $S_0[q] = \int_{t_0}^{t_1} dt\, L(q, \dot{q})$

dynamics determined by Euler-Lagrange derivatives $\frac{\delta L}{\delta q} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$

Jet-bundle of order 1: local coordinates $t, q, \dot{q}$

total derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \ddot{q}\frac{\partial}{\partial \dot{q}}$

Local functions : finite order in derivatives

Field theory

$E \rightarrow M$ $\quad J^\infty(E) \rightarrow M$ $\quad dim(M) = n$

$\phi^i, x^\mu$ $\quad \phi^i_{(\mu)} \equiv \phi^i_{\mu_1 \ldots \mu_k}, x^\mu$

total derivative $\partial_\nu = \frac{\partial}{\partial x^\nu} + \phi^i_{(\mu)\nu}\frac{\partial}{\partial \phi^i_{(\mu)}}$

Euler-Lagrange derivative $\frac{\delta}{\delta \phi^i} = (-)^{|\mu|}\partial_{(\mu)}\frac{\partial}{\partial \phi^i_{(\mu)}}$ $\quad \partial_{\mu_1 \ldots \mu_k} = \partial_{\mu_1} \ldots \partial_{\mu_k}$

In the case of classical mechanics, the action becomes a functional, and the dynamics of the system is determined by the Euler-Lagrange equations, which are ordinary differential equations.

When one computes the Euler-Lagrange derivative, one has to treat the derivative of q as an independent variable. This is the main idea of a jet-bundle, it is a space on which the fields and its derivatives are independent coordinates. The other remark is that the total derivative acting on a function which involves a given order of derivatives, say 1 for the Lagrangian in the example, involves the derivatives of one order higher. This is the reason why one has to go to the infinite jet-bundle containing all derivatives. Locality in this context means that the functions that one considers are all of a finite order in derivatives.

In field theory, the base space of the original bundle is of dimension n higher than one, local coordinates are denoted by $x^\mu$ and $\phi^i$, while the local coordinates on the jet-bundle involve the various derivatives. The same is true for the total derivative and the Euler-Lagrange derivative.

## Theory 2: Field theory Variational bicomplex

de Rham differential on jet-bundle
$$d = dx^\mu \frac{\partial}{\partial x^\mu} + d\phi^i_{(\mu)} \frac{\partial}{\partial \phi^i_{(\mu)}} = d_H + d_V$$

horizontal or total differential
$$d_H = dx^\mu \partial_\mu$$

vertical differential
$$d_V = (d\phi^i_{(\mu)} - dx^\nu \phi^i_{(\mu)\nu}) \frac{\partial}{\partial \phi^i_{(\mu)}}$$
"infinitesimal field variation"

bicomplex
$$(\Omega^{r,s}, d_H, d_V)$$

$$\omega^{r,s} = \frac{1}{r!s!} \omega_{\mu_1 \dots \mu_k i_1(\nu_1) \dots i_s(\nu_s)} dx^{\mu_1} \dots dx^{\mu_k} d_V \phi^{i_1}_{(\nu_1)} \dots d_V \phi^{i_s}_{(\nu_s)}$$

↖ local function

The de Rham differential on the infinite jet-bundle is split into two anticommuting differentials, the horizontal differential, involving the total derivative and the vertical one, corresponding to infinitesimal field variations.

As said before, the coefficients of the associated bicomplex are assumed to be local functions.

variational bicomplex

$$
\begin{array}{ccccccccccccccc}
 & & & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V & & \\
 & & 0 & \longrightarrow & \Omega^{0,3} & & & & \cdots & & \Omega^{n,3} & \xrightarrow{I} & \mathcal{F}^3 & \longrightarrow & 0 \\
 & & & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V & & \\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots\ \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{I} & \mathcal{F}^2 & \longrightarrow & 0 \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V & & \\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots\ \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{I} & \mathcal{F}^1 & \longrightarrow & 0 \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & & & \\
0 \longrightarrow & \mathbf{R} & & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots\ \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} & & & &
\end{array}
$$

local functional forms

$$\mathcal{F}^s = \Omega^{n,s}/d_H\Omega^{n-1,s}$$

Euler-Lagrange complex

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \quad (1.$$

locally exact

$$\xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \xrightarrow{\delta_V} \cdots$$

$$E = d_V\phi^i \frac{\delta}{\delta\phi^i} \qquad \omega^n = d_H\eta^{n-1} \iff \frac{\delta\omega^n}{\delta\phi^i} = 0$$

In the variational bicomplex, one adds a column to the previous bicomplex. Here the spaces FS are quotient spaces of top horizontal forms modulo exact ones. I is the projection and \delta_V is the differential induced by dV. The Euler-Lagrange complex is the complex obtained by going along the edge, with E the Euler-Lagrange derivative contracted with vertical generators. This complex is locally exact. Local exactness at the corner follows from the well-known fact that Euler-Lagrange derivatives annihilate total divergences and conversely, if the Euler-Lagrange derivative of a function vanishes, it is a total divergence.

## Globally

THEOREM 5.9. *The cohomology of the Euler-Lagrange complex* $\mathcal{E}^*(J^\infty(E))$

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,0}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,0}(J^\infty(E)) \cdots$$
$$\xrightarrow{d_H} \Omega^{n,0}(J^\infty(E)) \xrightarrow{E} \mathcal{F}^1(J^\infty(E)) \xrightarrow{\delta_V} \mathcal{F}^2(J^\infty(E)) \xrightarrow{\delta_V} \cdots \tag{5.25}$$

*is isomorphic to the de Rham cohomology of the total space* $E$, *that is*

$$H^p(\Omega^{*,0}(J^\infty(E)), d_H) \cong H^p(\Omega^*(E), d), \tag{5.26a}$$

*for* $p \leq n$, *and*

$$H^s(\mathcal{F}^*(J^\infty(E)), \delta_V) \cong H^p(\Omega^*(E), d), \tag{5.26b}$$

*for* $p = n + s$ *and* $s \geq 1$.

Anderson, The variational bicomplex

horizontal complex $\quad 0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{I} \mathcal{F}^0 \rightarrow 0$

The main global theorem in the subject is the result that the cohomology of the Euler-Lagrange complex is isomorphic to the cohomology of the bundle E. We will not be interested in the remainder of this talk by global aspects, but only consider what happens in a local coordinate patch.

Furthermore we will be interested in the horizontal part without vertical generators.

## Theory 2: Field theory Symmetries & Stationary surface

action functional $\mathcal{F}^0 \ni S_0 = [L d^n x] \cong \int_U (L d^n x)|_{\phi(x)}$

Lie algebra of symmetries $\mathcal{S} \ni \quad \delta_Q = \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}}, \quad \delta_Q [L d^n x] = [0] \Leftrightarrow \delta_Q L = \partial_\mu k^\mu$

$$[Q_1, Q_2]^i = \delta_{Q_1} Q_2^i - (1 \leftrightarrow 2)$$

stationary surface $\Sigma : \quad \partial_{(\mu)} \frac{\delta L}{\delta \phi^i} \approx 0$ NB: $det|\frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j}| \neq 0 \quad \Sigma : t, q^i, \dot{q}^i$

initial conditions

Noether operator $N^{+i} = N^{+i(\mu)} \partial_{(\mu)}, \quad N^{+i}(\frac{\delta L}{\delta \phi^i}) = 0$

associated symmetry $N^i = (-\partial)_{(\mu)} N^{+i(\mu)} \quad \delta_N [L d^n x] = [0]$

"gauge symmetries" $\mathcal{G} \subset \mathcal{S}$ Lie ideal

"global symmetries" $\mathcal{S}/\mathcal{G}$

The action in the field theory case is an element of F0, which is the algebraic version of a local functional. The idea is that if one considers the integral over a compact set U, one can freely integrate by parts, so that a local functional is determined by a horizontal n form up to exact ones.

A generalised symmetry is a vector field in the fiber that commutes with total derivative. As a consequence, it is determined by local functions Q^i. It furthermore has to leave the action functional invariant, or, in other words, the Lagrangian invariant up to a total divergence. Symmetries form a Lie algebra for the obvious bracket determined by the commutator of vector fields in terms of the characteristic functions Q^i.

The stationary surface is the surface defined in the jet-bundle by the Euler-Lagrange equations of L and its differential consequences. Note that, in the case of mechanics for instance, even if the Hessian associated to the Lagrangian is non degenerate, so that the EL equations can be written as second order differential equations in normal form, the stationary surface is still non trivial. Coordinates are given by the q^i and the first order time derivatives and Sigma is the space of initial conditions for the equations.

Noether operators are defined from identities among the left hand sides of these equations. By using integrations by parts, every Noether operator is easily seen to define a symmetry. Symmetries that are obtained in this way from Noether operators are called gauge symmetries. One can show that they form a Lie ideal in the space of all symmetries. The quotient Lie algebra is called the Lie algebra of global symmetries.

Theory 2: Field theory Longitudinal differential

(irreducible) generating set of Noether opertor

$$R_\alpha^{+i} = R_\alpha^{+i(\mu)}\partial_{(\mu)}, \quad R_\alpha^{+i}(\frac{\delta L}{\delta \phi^i}) = 0, \quad R_\alpha^{+i(\mu)} \not\approx 0$$

$$N^{+i}(\frac{\delta L}{\delta \phi^i}) = 0 \Rightarrow N^{+i} = Z^{+\alpha} \circ R_\alpha^{+i} + M^{+i}$$

trivial operators

$$M^{+i} = M^{+[j(\nu)i(\lambda)]}\partial_{(\nu)}\frac{\delta L}{\delta \phi^j}\partial_{(\lambda)}$$

irreducibility

$$Z^{+\alpha} \circ R_\alpha^{+i} \approx 0 \Rightarrow Z^{+\alpha} \approx 0$$

commutator is a gauge smmetry

$$\delta_{R_\alpha} R_\beta^i - (\alpha \leftrightarrow \beta) = Q_{\alpha\beta}^i$$

$$Q_{\alpha\beta}^i = (-\partial)_{(\mu)} Q_{\alpha\beta}^{+i(\mu)} \qquad Q_{\alpha\beta}^{+i} \approx f_{\alpha\beta}^{+\gamma} \circ R_\gamma^{+i}$$

additional generators $\partial_{(\mu)} C^\alpha$

longitudinal differential

$$\gamma = \delta_{R_\alpha(C^\alpha)} - \frac{1}{2}\partial_{(\mu)} f_{\alpha\beta}^\gamma (C^\alpha C^\beta)\frac{\partial}{\partial C_{(\mu)}^\gamma}, \qquad \gamma^2 \approx 0$$

What one needs now is a generating set of Noether identities, that allows to write any Noether identity in terms of them up to trivial operators that vanish when the equations hold and exist in any theory. A proper gauge theory is one for which the Noether operators are not all of this type. The generating set is irreducible if the set is independent when restricted to the stationary surface.

Since we know that gauge symmetries form an ideal, the commutator of the symmetries associated to a elements of the generating set gives a symmetry that comes from a Noether operator. It can thus be written in terms of the generating set itself. This defines the structure operators f+. Extending the bundle by additional generators and defining the operator \gamma through a similar formula than used in the finite-dimensional case, one finds that it is a differential on-shell due to the defining properties of the generating set.

## Theory 2: Field theory Koszul-Tate & BV

antifields $\partial_{(\mu)}\phi^*_i, \partial_{(\mu)}C^*_\alpha$ $\delta = \partial_{(\mu)}\frac{\delta L}{\delta\phi^i}\frac{\partial}{\partial\phi^*_{i(\mu)}} + \partial_{(\nu)}R^{+i\alpha}(\phi^*_i)\frac{\partial}{\partial C^*_{\alpha(\nu)}}$

resolution $(\Omega^{r,s}(\Sigma), d_H, d_V) \cong \quad H(\delta, (\Omega^{r,s,*}(E^A), d_H, d_V))$

HPT $s = \delta + \gamma + \dots, \qquad s^2 = 0 \qquad H(s, \Omega(E^{AC})) \cong H(\gamma, \Omega(\Sigma^C))$

antibracket $(\cdot,\cdot) : \mathcal{F}^{g_1} \times \mathcal{F}^{g_2} \longrightarrow \mathcal{F}^{g_1+g_2+1}$

$$(A, B) = [d^n x(\frac{\delta^R a}{\delta\phi^A}\frac{\delta^L b}{\delta\phi^*_A} - (\phi \leftrightarrow \phi^*))] \qquad A = [d^n x\, a], B = [d^n x\, b]$$

master equation $\frac{1}{2}(S, S) = 0$

$$S = [d^n x\,(L + \phi^*_i R^i_\alpha(C^\alpha) + \frac{1}{2}C^*_\gamma f^\gamma_{\alpha\beta}(C^\alpha C^\beta) + \dots)]$$

The Koszul-Tate resolution of Sigma is now defined by extending the fiber of the jet-bundle further through the approriate antifields and their derivatives. By suitably extending to the vertical generators, one gets a homological resolution of the bicomplex pulled bak to the stationary surface. The two differentials can again be combined into a single one, the BRST differential s using homological perturbation theory so that the cohomology of s reproduces the one of \gamma on shell.

The difference with the finite dimensional case is that the bracket is no longer a Poisson bracket because the space of functionals is no longer an algebra, just an odd graded Lie bracket. It is defined in terms of Euler-Lagrange derivatives of the representatives of the functionals.

The solution of the master equation looks as in the finite-dimensional case, up to the derivatives that are involved.

## Theory 2: Field theory Local BRST cohomology

generator in modified bracket $s = (S, \cdot)_{alt}$

$$(\cdot,\cdot)_{alt} : \mathcal{F} \times \Omega \to \Omega$$

$$(A, \cdot)_{alt} = \partial_{(\mu)} \frac{\delta^R a}{\delta \phi^i} \frac{\partial^L}{\partial \phi^*_{(\mu)}} - (\phi \leftrightarrow \phi^*)$$

local BRST cohomology $\{s, d_H\} = 0 \Rightarrow H(s, \mathcal{F})$ applications!

generated in standard antibracket $sA = (S, A)$

deformation theory in the space of local functionals

The BRST differential is generated in a modified bracket that does not involve Euler-Lagrange derivatives in the second argument, but acts as a graded derivation.

By construction, the BRST cohomology in the space of functions or forms that depend on antifields and ghosts reproduces the cohomology of the longitudinal differential. Because the BRST differential is defined so as to commute with the horizontal differential, it gives rise to a well defined differential in the space of local functionals. The associated cohomology groups are called local BRST cohomology groups. It is those cohomology groups that are involved in many applications.

In the space of local functionals, the BRST differential is generated in the standard antibracket, and the deformation theory sketched in the finite-dimensional case can thus be set-up in the space of local functionals scalar field theory

$$S = -[d^n x \, (\frac{1}{2}\,\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!}\phi^4)]$$

Yang-Mills theory

$$S = [d^n x \, (-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^{*\mu}_a D_\mu C^a + \frac{1}{2} C^*_c f^c_{ab} C^a C^b)]$$

general relativity

$$S = [d^n x \, (\sqrt{|g|} R + g^{*\mu\nu} \mathcal{L}_\xi g_{\mu\nu} - \xi^*_\mu \partial_\nu \xi^\mu)]$$

Poisson Sigma model

$$\begin{aligned} S_{\mathrm{BV}} \;=\; & \int_D \bar{\eta}_i \wedge d\bar{X}^i + \frac{1}{2}\alpha^{ij}(\bar{X})\bar{\eta}_i \wedge \bar{\eta}_j \\ & + \bar{X}^+_i \alpha^{ij}(\bar{X})\bar{\beta}_j + \bar{\eta}^{+i} \wedge (-d\bar{\beta}_i - \partial_i \alpha^{jk}(\bar{X})\bar{\eta}_j \bar{\beta}_k) \\ & - \frac{1}{2}\bar{\beta}^{+i}\partial_i \alpha^{jk}(\bar{X})\bar{\beta}_j \bar{\beta}_k - \frac{1}{4}\bar{\eta}^{+i} \wedge \bar{\eta}^{+j}\partial_i \partial_j \alpha^{kl}(\bar{X})\bar{\beta}_k \bar{\beta}_l \,. \end{aligned}$$

computation of $H(s, \mathcal{F})$ !

Let us finish this theoretical part by giving some examples. For the standard scalar field, the master equation coincides with the classical Lagrangian because all Noether operators are trivial ones. For Yang-Mills theory built with a semi-simple Lie algebra, the master equation is quite simple reflecting the fact that the standard generating set closes of shell with the structure constants of the Lie algebra involved. In general relativity, the non trivial gauge symmetries are given by Lie derivatives of the metric whose commutator involves structure operators with constant coefficients. For the Poisson sigma model the fact that the generating set of gauge symmetries closes only on-shell implies that solution of the master equation involves terms that are quadratic in the antifields.

What we have done in this context is the complete computation of the local BRST cohomology groups for the above three models. For the Poisson Sigma model, one can easily show, that locally, the cohomology is isomorphic to the cohomology of the Poisson cohomoloy in the target space.

## Applications I: Quantum field theory Perturbation theory

perturbative expansion of Green's functions

$$\frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}\phi \exp \frac{i}{\hbar}(I[\phi] + J_A\phi^A)}{\int \mathcal{D}\phi \exp \frac{i}{\hbar} I_0[\phi]} = \exp \frac{i}{\hbar} I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}] \exp \frac{i}{2\hbar} J_A(\mathcal{D}^{-1})^{AB} J_B.$$

free quadratic action

$$I_0 = -\tfrac{1}{2}\phi^A \mathcal{D}_{AB}\phi^B.$$

non trivial gauge invariance: not invertible because of zero eigenvalues

aim: make quadratic part invertible while still retaining consequences of gauge invariance

"gauge fixation"

generator of canonical transformation $\Psi[\phi]$

$$\begin{cases} \widetilde{\phi}^*_A = \phi^*_A + \dfrac{\delta^R \Psi}{\delta \phi^A} \\ \widetilde{\phi}^A = \phi^A \end{cases}$$

"gauge fixing fermion"

gauge fixed action

$$S_{gf}[\widetilde{\phi}, \widetilde{\phi}^*] = S[\widetilde{\phi}, \widetilde{\phi}^* + \frac{\delta \Psi}{\delta \phi}]$$

$$\frac{1}{2}(S_{gf}, S_{gf})_{\widetilde{\phi}, \widetilde{\phi}^*} = 0$$

In quantum field theory, the perturbative expansion of Green's functions involves inverting the quadratic part of the action. In theories with non trivial gauge invariance the quadratic part is not invertible because the matrix defining this quadratic part has zero eigenvalues.

The aim is then to fix, the gauge, i.e., to make the quadratic part invertible while still controlling the consequences of the gauge invariance of the original theory. In the BV formalism this is done as follows: one chooses a gauge fixing fermion, which is the generator of a canonical transformation that shifts only the antifields. Because we are considering a canonical transformation, the gauge fixed action still satisfies the master equation in the new variabes.

## Applications I: Quantum field theory — Anomalies

connected Green's functions $$W[J,\widetilde{\phi}^*] = \ln \frac{Z[J,\widetilde{\phi}^*]}{Z[0,\widetilde{\phi}^*]}$$

Legendre transform $$\widetilde{\phi}^{J,\widetilde{\phi}^*} = \frac{\delta W}{\delta J} \iff J = J^{\widetilde{\phi},\widetilde{\phi}^*}$$

effective action $$\Gamma[\widetilde{\phi},\widetilde{\phi}^*] = (W - J\phi)|_{J=J^{\widetilde{\phi},\widetilde{\phi}^*}} = S_{gf} + \hbar\Gamma^{(1)} + \dots$$

not a local functional

Zinn-Justin equation $$\frac{1}{2}(\Gamma,\Gamma) = \hbar A \circ \Gamma, \quad A \circ \Gamma = A + O(\hbar)$$

local functional

consistency condition $$(\Gamma,(\Gamma,\Gamma)) = 0 \Rightarrow (\Gamma, A\circ\Gamma) = 0 \Rightarrow (S,A) = 0$$

trivial anomalies absorbed through counterterm $$A = (S,B) \quad S \to S - \hbar B$$

nontrivial anomalies $$[A] \in H^1(s,\mathcal{F})$$

SU(3) YM theory $$TrC[d(AdA + \frac{1}{2}A^3)]$$ "Adler-Bardeen anomaly"

In the quantum theory, all information on the Green's functions is contained in the effective action, which is the Legendre transform of the log of the generating functional based on the gauge fixed action.

To lowest order, it is given by the gauge fixed action action itself, while the terms higher order in hbar are not local functionals any longer. One can then show that in renormlaized perturbation theory, the effective action does not satisfy the master equation, but is broken by terms of order hbar that start to lowest order with a local functional.

From the Jacobi identity for the antibracket, one then finds as a consistency condition that the lowest order contribution to the anomaly is a BRST cocycle. If it is exact, one can remove it to that order by adding a finite BRST breaking counterterm to the gauge fixed action. It follows that non trivial anomalies are classified by H1. By computing this group for YM theory based one SU(3), one finds for instance that the only element of this group is related to the the transgression in the Weil algebra relating Tr F^3 to Tr C^5 and given by the famous Adler–Bardeen anomaly.

## Applications I: Quantum field theory Counterterms

divergences in effective action $\Gamma^{(1)} = \frac{1}{\epsilon}\Gamma^{(1)-1} + finite$ ← local functional

consistency condition $\frac{1}{2}(\Gamma,\Gamma) = \hbar A \circ \Gamma \Rightarrow (S, \Gamma^{(1)-1}) = 0$

counterterm $S^{(1)} = S - \frac{\hbar}{\epsilon}\Gamma^{(1)-1}, \quad (S^{(1)}, S^{(1)}) = O(\hbar^2)$

trivial divergence $\Gamma^{(1)-1} = (S, \Xi)$ can be absorbed by canonical field antifield redefinition

renormalizability if $[\Gamma^{(1)-1}] \in H^0(s, \mathcal{F})$ can be absorbed by modifying coupling constants

4d semi-simple YM $H^0(s,\mathcal{F}) \cong [d^4x\, P]$

$P$ : group invariant polynomial in $Y^a_A = D_{\mu_1}\dots D_{\mu_k} F^a_{\nu\rho}$

consequence: $S_0 = [d^4x - \frac{1}{4g} F^a_{\mu\nu} F^{\mu\nu}_a]$ is renormalizable (powercounting, Lorentz invariance)

$S_0 = [d^4x P]$ renormalizable "in the modern sense"

When computing the perturbative expansion of the effective action, there arise divergent expressions in the one and higher loop contributions. One can show that the coefficient of this divergence is again a local functional. It then follows from the regularized Zinn-Justin equation that the coefficient of the divergence must be BRST closed.
It can thus be absorbed by a counterterm that is of the same form than the starting point action itself.

Furthermore, if it is exact, the divergence is trivial in the sense one does not need to add a new type of term to the action because it can be absorbed by a field-antifield redefinition generated by \Xi..
Non trivial divergences are thus controlled by H0. The theory is renormalizable, if every non trivial that counterterm actually arises, is already present in the action and can thus be absorbed by modifying the associated coupling constant. It is thus a stability requirement on the solution of the master equation.

In 4d semi-simple YM theory, one can show for instance that H0 is exhausted by group invariant polynomials in the covariant derivatives of the curvature. A a consequence, one finds, by using additional constraints such as powercounting and Lorentz invariance, that the standard YM action is renormalizable, i.e., that all non trivial countererms can be absorbed by the coupling constant of this original action. But one also immediately sees that if one takes as starting point action the most general invariant polynomial with independent coupling constants, not restricted for instance by power counting restrictions, the divergences can also be absorbed by modifying these coupling constants. This is what Weinberg calls renormalizability in the modern sense.

## Applications 2: NC field theory Seiberg-Witten map

non-commutative U(N) YM theory

$$\hat{S} = \int d^n x \,\mathrm{Tr}\,\left(-\frac{1}{4\kappa^2}\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu} + \hat{A}^{*\mu} * \hat{D}_\mu \hat{C} + \frac{1}{2}\hat{C}^* * [\hat{C} \,; \hat{C}]\right),$$

Weyl-Moyal star product

$$f * g(x) = \exp\left(i\wedge_{12}\right) f(x_1) g(x_2)|_{x_1=x_2=x}, \quad \wedge_{12} = \frac{\vartheta}{2}\theta^{\mu\nu}\partial^{x_1}_\mu \partial^{x_2}_\nu,$$

deformation of solution of master equation for standard Yang-Mills, controlled by

$$H^0(s,\mathcal{F}) \cong [d^4x\, P]$$

no antifield dependence

consequence:

$$\hat{S}[\hat{\phi}[\phi,\phi^*;\vartheta],\hat{\phi}_*[\phi,\phi^*;\vartheta];\vartheta] = S_0^{\mathrm{eff}}[\phi;\vartheta] + \sum_{k\geq 1} S_k^{(0)}[\phi,\phi^*],$$

$$S^{\mathrm{eff}}[A;\vartheta] =$$

$$= -\frac{1}{4\kappa^2}\int d^n x\,\mathrm{Tr}\,\left(F_{\mu\nu}F^{\mu\nu} + \frac{i\vartheta\theta^{\alpha\beta}}{2}(-F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + 4F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu})\right) + O(\vartheta^2).$$

"Seiberg-Witten map"

Non commutative U(N) Yang-Mills theory can be described by a solution of the master equation where the usual product is replaced by a Wey-Moyal star product. To order 0 in the deformation parameter \theta one thus recovers the usual solution of the master equation and non-commutative YM theory is thus a deformation of standard YM preserving the master equation. We have seen that this deformation theory is controlled by H0 and that H0 only contains invariant polynomials in the curvatures without any antifield dependence. It follows that there must exist a canonical field-antifield redefintion that allows to absorb the antifield dependent part of the deformation,i.e., that maps the gauge structure of the noncommutative to that of the commutative theory. This map is the so-called Seiberg-Witten map. These authors have deduced the existence of this map from string theory arguments. We see here that its existence can also more simply be understood as a consequence of the local BRST cohomology of YM theory.

## Applications 3: Classical field theory Consistent deformations

Start form free quadratic gauge theories

Construction of interactions preserving gauge invariance?

computation of $H^0(s,\mathcal{F})$ obstructions?

uniqueness results on YM construction or general relativity

massless spin 2 fields

$$S_0[h^a_{\mu\nu}] = \sum_{a=1}^N \int d^n x \left[ -\frac{1}{2} (\partial_\mu h^a{}_{\nu\rho}) (\partial^\mu h^{a\nu\rho}) + (\partial_\mu h^{a\mu}{}_\nu) (\partial_\rho h^{a\rho\nu}) - \left(\partial_\nu h^{a\mu}{}_\mu\right) (\partial_\rho h^{a\rho\nu}) + \frac{1}{2} (\partial_\mu h^{a\nu}{}_\nu) \left(\partial^\mu h^{a\rho}{}_\rho\right) \right], \quad n > 2.$$

gauge transformations

$$\delta_\epsilon h^a_{\mu\nu} = \partial_\mu \epsilon^a_\nu + \partial_\nu \epsilon^a_\mu$$

only possible deformation

$$S[g^a_{\mu\nu}] = \sum_a \frac{2}{\kappa_a^2} \int d^n x (R^a - 2\Lambda^a)\sqrt{-g^a}, \quad g^a_{\mu\nu} = \eta_{\mu\nu} + \kappa^a h^a_{\mu\nu},$$

The deformation theory explained before can actually be used quite effectively in order to construct consistent deformations, meaning interactions that preserve the gauge invariance. For that, one starts from a free gauge theories, computes the BRST cohomology of the free theory, and studies possible obstructions. In this way one prove a certain number of uniqueness results. For instance, the free theory corresponding to GR is the Pauli-Fierz theory describing massless spin 2 fields. If one takes several of these, several gravitons, and one tries to deform the solution of the master equation, one finds the the different gravitons cannot cross interact, contrary to what happens for spin 1 fields, but that one gets back, under suitable assumptions, disjoint copies of the Einstein-Hilbert action, as the only possible consistent deformation.

## Applications 3: Classical field theory Characteristic cohomology

standard techniques $$H^{-g}(s,\mathcal{F}) \cong H^{n-g}(d_H, \Omega^{*,0}(\Sigma))$$

"descent equations"

characteristic cohomology

consequence: characteristic cohomology for variational surface forms a graded Lie algebra

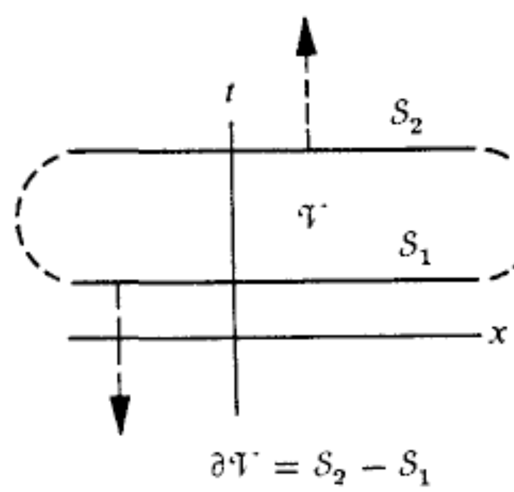
$g = 1$ $$H^{-1}(s,\mathcal{F}) \cong \mathcal{S}/\mathcal{G}$$ Lie algebra of global symmetries

$$H^{n-1}(d_H, \Omega^{*,0}(\Sigma)) \ni [j] \qquad \begin{cases} d_H j \approx 0, \\ j \sim j + d_H k + t, \quad t \approx 0 \end{cases}$$ conserved currents

canonical form for symmetry $$X^i \frac{\delta L}{\delta \phi^i} d^n x = d_H j_X$$

complete version of Noether's theorem that deals with ambiguities

Charges $$Q_X[\phi^s] = \oint_{\mathcal{S}} j_X[\phi^s]$$




Let us now turn to local BRST cohomology groups in negative ghost number. By standard diagram chasing techniques, called descent equations in this context, one can prove that the cohomology in negative ghost number is isomorphic to the cohomology of the horizontal complex, pulled back to the stationary surface \Sigma. the so-called characteristic cohomology of the surface Sigma. As a consequence, since the local BRST cohomology groups in negative ghost number forms a graded Lie algebra, one deduces that so does characteristic cohomology of a surface associated with partial differential equations that come from a Lagrangian.

In ghost number -1, the local BRST cohomology groups are esaily seen to be isomorphic to the Lie algebra of all all symmetries modulo gauge symmetries, those that come from Noether identities, which is the space we have called global symmetries. So what we have recovered is a complete version of Noether's theorem. Indeed, that there is correspondence between symmetries and conserved currents is well known and follows from the canonical form to which the definition of a symmetry can be brought to using integrations by parts. The question answered by our result is how to define the symmetries and the currents so that there is a 1-1 correspondence, i.e., how to take the ambiguities into account.

When evaluated for solutions, the charges defined by integration over an n-1 dimensional surface, do not depend on the representative of the current provided the surface is closed. This follows directly from Stokes's theorem. They then do not depend on the representative of the homology class of this surface either. The usual conservation in time of charges then follows by choosing a boundary of spacetime, for which the total charge vanishes, and assuming that the fields fall of sufficiently fast at spatial infinity, so that the charge over S1 equals the charge over S2.

# Applications 3: Classical field theory Characteristic cohomology

irreducible gauge theories (no 2,3-forms): $g \geqslant 3$ $H^{-g}(s, \mathcal{F}) \cong 0 \cong H^{n-g}(d_H, \Omega^{*,0}(\Sigma))$

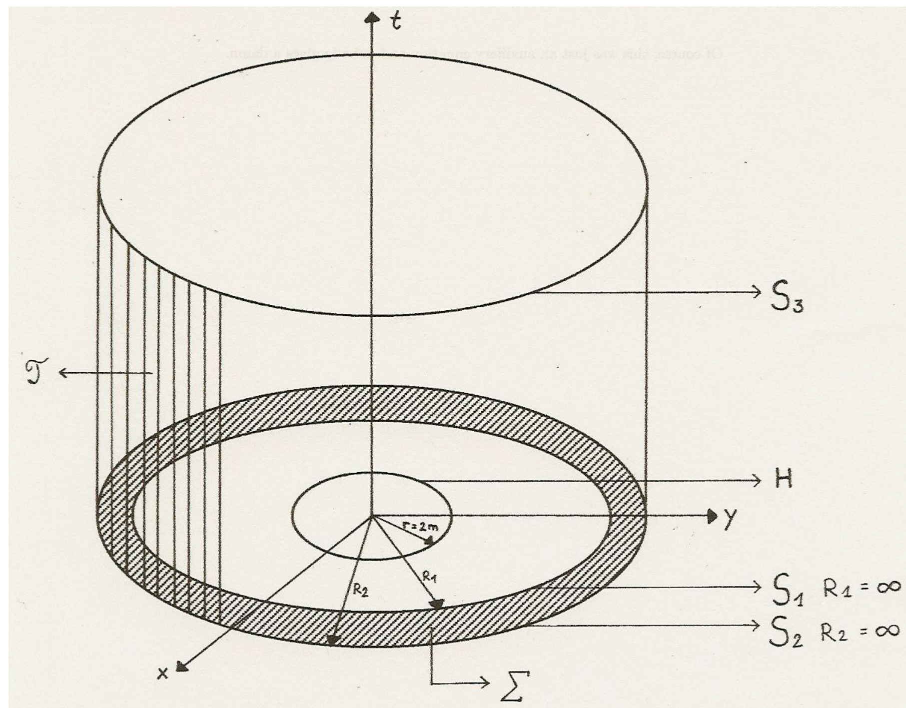
vanishing theorems for characteristic cohomology in low form degree

$g = 2$ $H^{-2}(s, \mathcal{F}) \ni [f^\alpha]$ $\begin{cases} R^i_\alpha(f^\alpha) \approx 0 \\ f^\alpha \sim f^\alpha + t^\alpha, t^\alpha \approx 0 \end{cases}$ "reducibility parameters"

associated conserved n-2 forms

$$[k_f^{n-2}] \in H^{n-2}(d_H, \Omega^{*,0}(\Sigma))$$

surface charges: $Q_f[\phi^s] = \oint_{S^{n-2}} k_f^{n-2}[\phi^s]$




For irreducible gauge theories, i.e., for gauge theories that do not contain 2,3-forms, one can quite easily show that all the cohomology groups in ghost number lower than -3 are trivial, which gives quite interesting vanishing theorems for characteristic cohomology in low form degrees.

In ghost number -2, the local BRST cohomology is determined by equivalence classes of gauge parameters that give on-shell vanishing gauge transformations. Parameters that vanish themselves on-shell should be considered as trivial. These reducibility parameters thus also classify the characteristic cohomology in degree n-2.

The associated charges are to be integrated over closed n-2 dimensional surfaces, for example spheres at constant t and r. By applying Stokes' theorem, they do not depend on t nor on r, which is what makes them interesting in physics.

## Applications 3: Classical field theory Characteristic cohomology

derived bracket: $K \in H^0 : \quad \frac{1}{2}(K, K) = 0 \qquad H^{-3} = 0$

$$F \in H^{-2}, \quad G_F = (F, K) \in H^{-1}$$

$\Longrightarrow \quad H^{-2}$ is a Lie algebra with bracket $\quad [F_1, F_2] = (G_{F_1}, F_2)$

Finally, the following construction will be useful. Let K be an element of H0 that defines a deformation with vanishing first obstruction. Let H-3 be zero and consider the element GF of H-1 obtained by taking the bracket of an element F of H2 with K. Then H-2 is a Lie algebra for the induced bracket.i.e, the bracket obtained by acting with GF1 on F2.

## Applications 3: Classical field theory — Surface charges

### Examples

semi-simple YM theory: $\delta_\epsilon A^a_\mu = D_\mu \epsilon^a = 0 \Longrightarrow \epsilon^a = 0$

EM: $\delta_\epsilon A_\mu = \partial_\mu \epsilon = 0 \Longrightarrow \epsilon = cte \quad \longleftrightarrow \quad k^{n-2} = {}^*F$

electric charge $Q = \oint_{S^{n-2}} {}^*F$

GR: $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0 \Longrightarrow \xi^\mu = 0$

linearized gravity: $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \xi^\mu$ Killing vector of $\bar{g}_{\mu\nu}$

$$k_\xi[h;\bar{g}] = \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}\sqrt{-\bar{g}}\Big(\bar{\xi}^\nu \bar{D}^\mu h + \bar{\xi}^\mu \bar{D}_\sigma h^{\sigma\nu} + \bar{\xi}_\sigma \bar{D}^\nu h^{\sigma\mu} + \frac{1}{2} h \bar{D}^\nu \bar{\xi}^\mu + \frac{1}{2} h^{\mu\sigma} \bar{D}_\sigma \bar{\xi}^\nu + \frac{1}{2} h^{\nu\sigma} \bar{D}^\mu \bar{\xi}_\sigma - (\mu \longleftrightarrow \nu)\Big),$$

$$Q_\xi = \oint_{S^{n-2}} k_\xi[h, \bar{g}]$$

application: first law of BH mechanics

$$\oint_{S^\infty} k_{\frac{\partial}{\partial t}} = \oint_H k_{\frac{\partial}{\partial t}}$$

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A}$$

For example, in YM theory they would be defined by gauge parameters whose covariant derivatives vanish, for all choices of potentials. All such parameters vanish for semi-simple gauge groups, however in the U(1) case, for electromagnetism, there is precisely 1 non-vanishing solution, a constant gauge parameter. The associated conserved n-2 form turns out to be precisely the dual of the field strength F, so that the charge is of course the electric charge, as determined by Gauss's law.

In GR, one would have to find Killing vectors for a generic metric, which do not exist, so there are again no non trivial surface charges. For linearized gravity however, the gauge transformations of the metric perturbations involve the Lie derivative of the background metric, which might very well have Killing vectors, for instance in the flat case, the Kvf represent the Poincare algebra. The associated surface charges can then be constructed. In a flat background, they describe the ADM energy-momentum and angular momentum. In a curved background, they coincide with expressions derived originally in the AdS context by Abott and Deser.

As an application in GR, for a stationary black hole solution and small perturbations around it, it is the linearized theory around the BH that is relevant. One considers the conserved form associated to the time-like Killing vector. The first law of black hole mechanics follows from the r independence of these forms. Indeed, when evaluating at infinity, one recovers by definition the variation of the mass, while at the horizon, one can show that it reduces to the surface gravity times the variation of the area.

## Applications 3: Classical field theory — Surface charges

expand GR $S_{GR} = S^2 + S^3 + \dots$

global symmetry $\mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \quad \delta^1_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}$

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ Poincaré invariance of Pauli-Fierz theory

derived bracket: Lie bracket of Killing vector fields

Surface charges form a representation of the algebra of Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta^1_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]}$$

full GR, asymptotics

$g_{\mu\nu} = \bar{g}_{\mu\nu} + O(\frac{1}{r^{\chi_{\mu\nu}}})$ at boundary $r \longrightarrow \infty$

replace $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ charges $Q_\xi = \oint_{S^\infty} k_\xi[g - \bar{g}, \bar{g}]$

When expanding the full master action of GR in term of the homogeneity in the fields, we know that the cubic term defines a deformation of the free theory that is unobstructed and thus can be used to construct the derived bracket in the space of Killing vectors of the background metric, since we know also that H3 is 0, as we deal with an irreducible gauge theory. The associated element of H-1 is a global symmetry which corresponds to taking the Lie derivative of the metric perturbation. If we expand around the flat background, we recover the global Poincare invariance of Pauli-Fierz theory. The derived bracket in H2 reduces to the Lie bracket for Killing vector fields. Because of the isomorphism, we find that the surface charges form a Lie algebra that is isomorphic to that of the Killing vector fields. Explicitly, the bracket is obtained by acting with the induced global symmetry.

The question is then how to use this analysis of the linearized theory in full GR.

One application is in the case where the metric approaches a background metric with some appropriate fall-off conditions. What one does is replace the metric perturbations by the difference between between the full metric and the background. In the asymptotic region, where the linearized EOM are supposed to be valid due to the fall-off conditions, one then still has conservation in time, but not in r because the linearized approximation will fail as one goes into the bulk. One way to think about this is that the linearized surface integrals contain all the sources, including those due to the self-interactions of the fields, only in the asymptotic region.

## Applications 3: Classical field theory Algebra & asymptotics

new feature: asymptotic Killing vectors $\mathcal{L}_\xi \bar{g}_{\mu\nu} \to 0$ to leading order

that preserve the fall-off conditions $\mathcal{L}_\xi g_{\mu\nu} = O(\frac{1}{r^{\chi_{\mu\nu}}})$

suitable tuning of fall-off conditions on metrics and asymptotic Killing vectors:

centrally extended charge representation of algebra of asymptotic Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]} + K_{\xi_1,\xi_2} \qquad K_{\xi_1,\xi_2} = \oint_{S^\infty} k_{\xi_2}[\mathcal{L}_{\xi_1}\bar{g}, \bar{g}]$$

NB: central extension vanishes for exact symmetries of the background

The new feature is then that one will only need approximate Killing vectors in order to guarantee conservation of charges. One has to require that the associated large gauge transformations leave the space of asymptotically background metrics invariant. Under suitable fall-off conditions on metrics and gauge parameters, one then arrives at the following algebra of conserved charges, which represents the algebra of asymptotic reducibility parameters. The central extension vanishes for exact Killing vector fields of the background metric.

## Applications 3: Classical field theory Asymptotically AdS

non trivial asymptotic Kvf= conformal Kvf of flat boundary metric

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1+\frac{r^2}{l^2})d\tau^2 + +(1+\frac{r^2}{l^2})^{-1}dr^2 + r^2\sum_A f_A(dy^A)^2,$$

n>3: $\mathfrak{so}(n-1,2)$ only exact Killing vectors of AdS, no central extension

n=3: pseudo-conformal algebra in 2 dimensions, 2 copies of Wit algebra

charge algebra: 2 copies of Virasoro

$$i\{\mathcal{L}^\pm_m, \mathcal{L}^\pm_n\} = (m-n)\mathcal{L}^\pm_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m},$$

$$\{\mathcal{L}^\pm_m, \mathcal{L}^\mp_n\} = 0,$$

cornerstone of AdS3/CFT2 correspondence

where $c = \frac{3l}{2G}$ is the central charge for the anti-de Sitter case.

similar results in de Sitter spacetimes at timelike infinity

In the example of asymptotically AdS spacetimes, one then finds that the non trivial asymptotic Killing vectors are given by the conformal Killing vectors of the flat metric induced on the boundary.

For n> 3, one finds so(n-1,2), the exact Kvf of the background and thus no central extension.

In three dimensions, one finds the pseudo-conformal algebra in 2 dimensions. For the charges one then gets 2 copies of the centrally extended Virasoro algebra, with a central charge that involves the gravitational coupling G and the cosmological constant l. This result, obtained in the mid-eighties, is a cornerstone of the AdS3/CFT2 correspondence because it gives concrete evidence that a gravity theory in three dimensions could be described by a two dimensional conformal field theory at its boundary.

In the de Sitter case, essentially the same results hold, except that the boundary is now at timelike infinity instead of spatial infinity.

conformal boundary in asymptotically flat spacetimes: null infinity

Introducing the retarded time $u = t - r$, the luminosity distance $r$ and angles $\theta^A$ on the $(n-2)$-sphere by $x^1 = r\cos\theta^1$, $x^A = r\sin\theta^1\cdots\sin\theta^{A-1}\cos\theta^A$, for $A = 2, \ldots, n-2$, and $x^{n-1} = r\sin\theta^1\cdots\sin\theta^{n-2}$, the Minkowski metric is given by

$$d\bar{s}^2 = -du^2 - 2\,du\,dr + r^2\sum_{A=1}^{n-2} s_A (d\theta^A)^2, \tag{3.1}$$

where $s_1 = 1$, $s_A = \sin^2\theta^1\cdots\sin^2\theta^{A-1}$ for $2 \leqslant A \leqslant n-2$. The (future) null boundary is defined by $r = \text{constant} \to \infty$ with $u, \theta^A$ held fixed.

$$\mathfrak{bms}_n$$

$$\begin{aligned} \xi^u &= T(\theta^A) + u\partial_1 Y^1(\theta^A) + o(r^0),\\ \xi^r &= -r\partial_1 Y^1(\theta^A) + o(r),\\ \xi^A &= Y^A(\theta^B) + o(r^0), \qquad A = 1, \ldots, n-2, \end{aligned}$$

$Y^A(\theta^A)$ conformal Kvf of n-2 sphere

$T(\theta^A)$ "supertranslations", arbitrary function on n-2 sphere

Witten suggested in 2001 that the appropriate boundary from a conformal point of view in asymptotically flat spacetimes is null infinity. If one introduces the retarded time u, future null infinity is still at r goes to infinity for fixed u and fixed angles.

The non trivial asymptotic Killing vectors turn out to be determined by functions T and Y which depend on the angles. The functions Y describe conformal Killing vectors of the n-2 sphere, while the functions T depend arbitrarily on the angles and are the so-called supertranslations.

$$\widehat{\xi} = [\xi, \xi']$$

$$\hat{T} = Y^A \partial_A T' + T \partial_1 Y'^1 - Y'^A \partial_A T - T' \partial_1 Y^1,$$
$$\hat{Y}^A = Y^B \partial_B Y'^A - Y'^B \partial_B Y^A.$$

algebra: semi-direct product with abelian ideal $\mathfrak{i}_{n-2}$

n>4: $\mathfrak{so}(n-1,1) \ltimes \mathfrak{i}_{n-2}$

n=4: conformal algebra in 2d $\ltimes \mathfrak{i}_2$

$\cup$

$\mathfrak{so}(3,1)$ Bondi-Metzner-Sachs (1962)

As an algebra, one finds the semi-direct product of the conformal Kvf's of the n-2 sphere with the abelian ideal of supertranslations.

In 4 dimensions, one finds the semi-direct product of the 2d conformal algebra with the supertranslations. The former contains the Lorentz algebra as a subalgebra. In the original 1962 derivation by Bondi Metzner & Sachs of the symmetry group of asymptotically flat spacetimes at null infinity, they required these trsf to be well defined on the 2-sphere, and found thus only the Lorentz algebra and as only symmetry enhancement with respect to the exact case the supertranslations, which contain the ordinary translations for particular choices of the function T. It would be interesting to study if there are central extensions in the representation by charges of bms4. We have not done so, though.

## Central extensions: Asymptotically flat spacetimes

n=3: no restriction on $Y(\theta)$

$$J_n = \xi(T=0, Y=\exp(in\theta))$$

$$P_n \equiv \xi(T=\exp(in\theta), Y=0)$$

$$\begin{aligned} i[J_m, J_n] &= (m-n)J_{m+n}, \\ i[P_m, P_n] &= 0, \\ i[J_m, P_n] &= (m-n)P_{m+n}. \end{aligned}$$

1 copy of Wit algebra acting on $\mathfrak{i}_1$

$$\cup$$

$$\mathfrak{iso}(2,1)$$

Ashtekar et al. (1997)

charge algebra:

$$\begin{aligned} i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\mathcal{J}_{m+n}, \\ i\{\mathcal{P}_m, \mathcal{P}_n\} &= 0, \\ i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\mathcal{P}_{m+n} + \frac{1}{4G}m(m^2-1)\delta_{n+m}. \end{aligned}$$

relation to $AdS_3$ similar to contraction between $\mathfrak{so}(2,2) \to \mathfrak{iso}(2,1)$

$$\begin{aligned} i[J_m, J_n] &= (m-n)J_{m+n}, \\ i[P_m, P_n] &= \frac{1}{l^2}(m-n)J_{m+n}, \\ i[J_m, P_n] &= (m-n)P_{m+n}, \end{aligned}$$

$$L^{\pm}_m = \frac{1}{2}(lP_{\pm m} \pm J_{\pm m}) \qquad l \to \infty$$

In 3 dimensions, the conformal Killing equation on the circle imposes no restrictions on the function Y(theta), so that the algebra is described by 2 arbitrary functions on the circle. After Fourier analyzing, the algebra consists of 1 copy of the Wit algebra acting on the functions on the circle in a similar way than the Lorentz transformations act on the ordinary translations. In fact the bms3 algebra has been originally discussed in a paper by Ashtekar et al in 1997.

What was not done though was the computation of the charge algebra. It turns out to contain a non trivial central extension between the two factors. A posteriori, it is clear that this is the only place that the central extension as it cannot appear in the one copy of the Wit algebra on account of the missing dimensional parameter l in the flat case. The relation to the AdS3 Virasoro case is by a contraction similar to the one between so(2,2) and iso(2,1). More precisely, if one introduces a parameter of dimension length, there is an extension of the BMS algebra, and after redefining the generators, one finds both for the asymptotic symmetries and for the charges, including the central ones, the AdS3 results.

What would be interesting is to analyze in details what this classical central extension can teach us about quantum gravity in asymptotically flat 3d spacetimes. Indeed, let me recall that in the AdS3 case, the central extension was a crucial ingredient that allowed Strominger to use the Cardy formula to give a microscopic explanation of black entropy.

# Selected references

## Reviews on BV


M. Henneaux, "Hamiltonian Form Of The Path Integral For Theories With A Gauge Freedom," Phys. Rept. **126** (1985) 1.

M. Henneaux and C. Teitelboim, "Quantization of gauge systems," *Princeton, USA: Univ. Pr. (1992) 520 p.*

J. Gomis, J. París and S. Samuel, "Antibracket, antifields and gauge theory quantization," Phys. Rept. **259** (1995) 1 [hep-th/9412228].

O. Piguet and S.P. Sorella, "Algebraic renormalization: Perturbative renormalization, symmetries and anomalies," *Berlin, Germany: Springer (1995) 134 p. (Lecture notes in physics: m28).*

S. Weinberg. The Quantum Theory of Fields. vol. 2 : Modern Applications. Cambridge, UK : Univ. Pr. (1996) 489 p.


To conclude, here is a list of selected references on the topics I have sketched during this review. If people are interested in more details in any one of these topics, feel free to contact me, I am here all week ...

## Locality, jet-bundles

I. Anderson, "The variational bicomplex," tech. rep., Formal Geometry and Mathematical Physics, Department of Mathematics, Utah State University, 1989.

I. Anderson, "Introduction to the variational bicomplex," in *Mathematical Aspects of Classical Field Theory*, M. Gotay, J. Marsden, and V. Moncrief, eds., vol. 132 of *Contemporary Mathematics*, pp. 51–73. Amer. Math. Soc., 1992.

P. Olver, *Applications of Lie Groups to Differential Equations*. Spinger Verlag, New York, 2nd ed., 1993. 1st ed., 1986.

M. Henneaux, "Space-Time Locality of the BRST Formalism," Commun. Math. Phys. **140** (1991) 1.

G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in gauge theories," *Phys. Rept.* **338** (2000) 439–569, `hep-th/0002245`.

## Deformation theory & BV

G. Barnich and M. Henneaux, "Consistent couplings between fields with a gauge freedom and deformations of the master equation," Phys. Lett. **B311** (1993) 123 [hep-th/9304057].

# Asymptotic symmetries in gravity


H. Bondi, M. G. van der Burg, and A. W. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond. A* **269** (1962) 21.

R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond. A* **270** (1962) 103.

R. K. Sachs, "Asymptotic symmetries in gravitational theory," *Phys. Rev.* **128** (1962) 2851.

J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Commun. Math. Phys.* **31** (1973) 161–170.

L. F. Abbott and S. Deser, "Stability of gravity with a cosmological constant," *Nucl. Phys.* **B195** (1982) 76.

L. F. Abbott and S. Deser, "Charge definition in nonabelian gauge theories," *Phys. Lett.* **B116** (1982) 259.

M. Henneaux and C. Teitelboim, "Asymptotically anti-de Sitter spaces," *Commun. Math. Phys.* **98** (1985) 391.

J. D. Brown and M. Henneaux, "On the Poisson brackets of differentiable generators in classical field theory," *J. Math. Phys.* **27** (1986) 489.

J. D. Brown and M. Henneaux, "Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity," *Commun. Math. Phys.* **104** (1986) 207.

R. M. Wald, "Black hole entropy is Noether charge," *Phys. Rev.* **D48** (1993) 3427–3431, gr-qc/9307038.

V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," *Phys. Rev.* **D50** (1994) 846–864, gr-qc/9403028.

I. M. Anderson and C. G. Torre, "Asymptotic conservation laws in field theory," *Phys. Rev. Lett.* **77** (1996) 4109–4113, hep-th/9608008.

G. Barnich and F. Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," *Nucl. Phys.* **B633** (2002) 3–82, hep-th/0111246.

G. Barnich and G. Compère, "Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions," *Class. Quant. Grav.* **24** (2007) F15, gr-qc/0610130.

## Origin, QFT


C. Becchi, A. Rouet and R. Stora, "The Abelian Higgs-Kibble Model. Unitarity Of The S Operator," Phys. Lett. **B52** (1974) 344.

C. Becchi, A. Rouet and R. Stora, "Renormalization Of The Abelian Higgs-Kibble Model," Commun. Math. Phys. **42** (1975) 127.

C. Becchi, A. Rouet and R. Stora, "Renormalization Of Gauge Theories," Annals Phys. **98** (1976) 287.

C. Becchi, A. Rouet and R. Stora, "Renormalizable Theories With Symmetry Breaking", in "Field Theory Quantization and Statistical Physics", E. Tirapegui ed., Reidel, Dordrecht 1981 (preprint dated 1975).

I.V. Tyutin, "Gauge Invariance In Field Theory And Statistical Physics In Operator Formalism," LEBEDEV-75-39.

J. Zinn-Justin, "Renormalization Of Gauge Theories," *Lectures given at Int. Summer Inst. for Theoretical Physics, Jul 29 - Aug 9, 1974, Bonn, West Germany* published in *Trends in Elementary Particle Physics*, Lectures Notes in Physics 37, H. Rollnik and K. Dietz eds. (Springer Verlag, Berlin 1975).

J. Zinn-Justin, "Quantum Field Theory And Critical Phenomena," *Oxford, UK: Clarendon (1989), 914 p. (International series of monographs on physics, 77).* Third ed. 1996, chapter 21.

I.A. Batalin and G.A. Vilkovisky, "Relativistic S Matrix Of Dynamical Systems With Boson And Fermion Constraints," Phys. Lett. **69B** (1977) 309.

I.A. Batalin and G.A. Vilkovisky, "Gauge Algebra And Quantization," Phys. Lett. **102B** (1981) 27.

I.A. Batalin and G. A. Vilkovisky, "Feynman Rules For Reducible Gauge Theories," Phys. Lett. **B120** (1983) 166.

I.A. Batalin and G.A. Vilkovisky, "Quantization Of Gauge Theories With Linearly Dependent Generators," Phys. Rev. **D28** (1983) 2567.

I.A. Batalin and E.S. Fradkin, "A Generalized Canonical Formalism And Quantization Of Reducible Gauge Theories," Phys. Lett. **122B** (1983) 157.

I.A. Batalin and G.A. Vilkovisky, "Closure Of The Gauge Algebra, Generalized Lie Equations And Feynman Rules," Nucl. Phys. **B234** (1984) 106.

I.A. Batalin and G.A. Vilkovisky, "Existence Theorem For Gauge Algebra," J. Math. Phys. **26** (1985) 172.

B. L. Voronov, P. M. Lavrov and I. V. Tyutin, "Canonical Transformations And The Gauge Dependence In General Gauge Theories ," Sov. J. Nucl. Phys. **36** (1982) 292.

E. Witten, "A Note On The Antibracket Formalism," Mod. Phys. Lett. **A5** (1990) 487.

J. Gomis and S. Weinberg, "Are Nonrenormalizable Gauge Theories Renormalizable?," Nucl. Phys. **B469** (1996) 473 [hep-th/9510087].