Infinite dimensional Lie algebras beyond Kac-Moody and Virasoro algebras

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0.Abelian extensions of Lie algebras

Let $\mathbf{0} \to \mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to \mathbf{0}$ be a topologically split short exact sequence of locally convex Lie algebras; $\mathfrak{a}$ abelian
(an abelian extension of $\mathfrak{g}$ by $\mathfrak{a}$). Then
$\mathfrak{a}$ carries a $\mathfrak{g}$-module structure; $(x, a) \mapsto x.a$.

Let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ cont. lin. section of $q \Rightarrow$

$\omega\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a}, \quad \omega(x,y) := [\sigma(x), \sigma(y)] - \sigma([x,y])$.

Then ω is a cont. 2-cocycle:

$$\sum_{cyc.} x.\omega(y,z) - \omega([x,y],z) = 0$$

and $\widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g}$ with the bracket

$[(a,x),(a',x')] = (x.a' - x'.a + \omega(x,x'), [x,x'])$.

$$\mathrm{Ext}(\mathfrak{g},\mathfrak{a}) \cong H^2(\mathfrak{g},\mathfrak{a}) := Z^2(\mathfrak{g},\mathfrak{a})/B^2(\mathfrak{g},\mathfrak{a}),$$

(2nd Lie algebra cohomology)

where $Z^2(\mathfrak{g},\mathfrak{a})$ (cont. 2-cocycles) and

$B^2(\mathfrak{g},\mathfrak{a}) = \{\omega(x,y) = \ell([x,y]), \ell \in \mathrm{Hom}(\mathfrak{g},\mathfrak{a})\}$.

1. The classical algebras

$\mathfrak{k}$ simple fin. dim. complex Lie algebra
Loop algebra: $\mathcal{L}(\mathfrak{k}) := C^\infty(\mathbb{S}^1, \mathfrak{k})$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$
$[f,g](t) := [f(t), g(t)]$.

Universal central extension $\widehat{\mathcal{L}}(\mathfrak{k}) = \mathbb{C} \oplus_\omega \mathcal{L}(\mathfrak{k})$ with cocycle

$$\omega(f,g) = \int_0^1 \kappa(f, g')\, dt = \int_{\mathbb{S}^1} \kappa(f, dg),$$

(κ is the Cartan–Killing form).

Affine Kac–Moody algebra (untwisted):
$\widehat{\mathcal{L}}(\mathfrak{k}) \rtimes \mathbb{C}\frac{d}{dt}$.

Virasoro alg.: $\mathfrak{vir} = \mathbb{C} \oplus_\eta \mathcal{V}(\mathbb{S}^1)_\mathbb{C}$ (cent. ext.)

$\eta(f\frac{d}{dt}, g\frac{d}{dt}) = \int_0^1 f''g' - g''f'\, dt.$

Both (affine Kac-Moody and Virasoro) are contained in a central extension:

$$\mathbb{C} \oplus_{\omega+\eta} \Big(\mathcal{L}(\mathfrak{k}) \rtimes \mathcal{V}(\mathbb{S}^1)_\mathbb{C}\Big).$$

What is $H^2(\mathcal{L}(\mathfrak{k}) \rtimes \mathcal{V}(\mathbb{S}^1)_\mathbb{C}, \mathbb{C})$?

Goals and Problems

M: a compact connected smooth manifold
$\mathcal{V}(M)$: smooth vector fields on M
$\mathfrak{k}$ a finite-dim. Lie algebra (not nec. simple)

Problem 1: Determine all central extensions of $C^\infty(M,\mathfrak{k})$.

Problem 2: Determine $\mathcal{V}(M)$-covariant central extensions $V \oplus_\omega C^\infty(M,\mathfrak{k})$ by natural $\mathcal{V}(M)$-modules V. Are there natural universal extensions of this type?

Problem 3: Determine the "twists" of the extensions $(V \oplus_\omega C^\infty(M,\mathfrak{k})) \rtimes \mathcal{V}(M)$, i.e., the space $H^2(\mathcal{V}(M), V)$.

Problem 4: Which of these extensions are integrable to Lie group extensions?

Problem 5: Can we do the same for $\mathfrak{aut}(P)$, where $q\colon P \to M$ is a principal K-bundle? (Trivial case: $\mathfrak{aut}(M \times K) \cong C^\infty(M,\mathfrak{k}) \rtimes \mathcal{V}(M)$)

2. Abelian extensions of semidirect sums

$\mathfrak{h} = \mathfrak{n} \rtimes \mathfrak{g}$ topological Lie algebra, $q\colon \mathfrak{h} \to \mathfrak{g}$.
V a topological $\mathfrak{h}$-module,
$V^{\mathfrak{n}}$ ($\mathfrak{n}$-invariants in V) is a closed $\mathfrak{h}$-submodule

Inflation: $I\colon H^2(\mathfrak{g}, V^{\mathfrak{n}}) \to H^2(\mathfrak{h}, V), [\omega] \mapsto [q^*\omega]$

Restrictions: $R_{\mathfrak{g}}\colon H^2(\mathfrak{h}, V) \to H^2(\mathfrak{g}, V)$
$R_{\mathfrak{n}}\colon H^2(\mathfrak{h}, V) \to H^2(\mathfrak{n}, V)^{[\mathfrak{g}]}$

$H^2(\mathfrak{n}, V)^{[\mathfrak{g}]}$ consists of those classes $[f]$ for which there exista a cont. bilinear $\theta\colon \mathfrak{g}\times\mathfrak{n} \to V$ with $x.f = \mathrm{d}_{\mathfrak{n}}(\theta(x))$ for $x \in \mathfrak{g}$. Equivalently:

$$x.(v, n) := (x.v + \theta(x)(n), x.n)$$

defines an action of $\mathfrak{g}$ on $\widehat{\mathfrak{n}} = V \oplus_f \mathfrak{n}$ and $\widehat{\mathfrak{n}} \rtimes \mathfrak{g}$ is an extension of $\mathfrak{n} \rtimes \mathfrak{g}$ by V.

Thm. If $\mathfrak{n}$ is perfect and $V = V^{\mathfrak{n}}$, then

$$(R_{\mathfrak{n}}, R_{\mathfrak{g}})\colon H^2(\mathfrak{h}, V) \to H^2(\mathfrak{n}, V)^{[\mathfrak{g}]} \oplus H^2(\mathfrak{g}, V)$$

is a linear isomorphism.

3. Central extensions of $C^\infty(M,\mathfrak{k})$

$\mathfrak{k}$ fin. dim. Lie alg., $\mathfrak{g} = C^\infty(M,\mathfrak{k})$
$\kappa \in \mathrm{Sym}^2(\mathfrak{k},V)^{\mathfrak{k}}$ inv. symm. bilinear
$\eta \in C^2(\mathfrak{k},V) = \mathrm{Alt}^2(\mathfrak{k},V)$ 2-cochain
Three fundamental types of cocycles:

(I) $\omega_\kappa(f,g) := [\kappa(f,\mathtt{d}g)]$
values in $\overline{\Omega}^1(M,V) := \Omega^1(M,V)/\mathtt{d}C^\infty(M,V)$

(II) $\omega_\eta(f,g) = \eta(f,g) \in C^\infty(M,V)$; $\mathtt{d}_{\mathfrak{k}}\eta = 0$.

(III) $\omega_{\kappa,\eta}(f,g) = \kappa(f,\mathtt{d}g) - \kappa(g,\mathtt{d}f) - \mathtt{d}(\eta(f,g))$
values in $\Omega^1(M,V)$ if $\mathtt{d}\eta = C(\kappa)$, where
$C(\kappa)(x,y,z) = \kappa([x,y],z)$ is assoc. 3-cocycle

Thm. (N., Wagemann, '05;
based on Haddi '92, Zusmanovich '94)
For $\mathfrak{k}$ perfect and $\omega \in Z^2(\mathfrak{g},\mathbb{R})$, there exist
$\kappa_i \in \mathrm{Sym}^2(\mathfrak{k},V_i)^{\mathfrak{k}}$, $\eta_i \in C^2(\mathfrak{k},V_i)$, $i = 1,2,3$,
and
$\beta_1 \in \overline{\Omega}^1(M,V_1)'$ (a closed 1-current),
$\beta_2 \in C^\infty(M,V_2)'$ (a distribution),
$\beta_3 \in \Omega^1(M,V_3)'$ (a 1-current) with

$$[\omega] = [\beta_1 \circ \omega_{\kappa_1} + \beta_2 \circ \omega_{\eta_2} + \beta_3 \circ \omega_{\kappa_3,\eta_3}] \in H^2(\mathfrak{g},\mathbb{R}).$$

More on the fundamental cocycles

• All coboundaries in $B^2(\mathfrak{g}, \mathbb{R})$ are of type (II): $\beta \circ \omega_\eta$, $\beta \in C^\infty(M, \mathfrak{k})'$, $\eta(f,g) = [f,g]$.

• Type (II) is needed iff $H^2(\mathfrak{k}, \mathbb{R}) \neq \mathbf{0}$.

• Type (III) is needed iff $\exists \kappa \in \mathrm{Sym}^2(\mathfrak{k}, \mathbb{R})^{\mathfrak{k}}$ such that $C(\kappa)$ is non-zero 3-coboundary.

• $[C(\kappa)] = 0 \Rightarrow \kappa$ vanishes on Levi subalgs; Converse is false: $\exists$ 50-dim. counterex. (Angelopoulos/Benayadi '93).

• If $\mathfrak{k}$ is semisimple, then type (I) suffices. If $\kappa_u \colon \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is universal, then $\omega_{\kappa_u} \in Z^2(\mathfrak{g}, \overline{\Omega}^1(M, V(\mathfrak{k}))$ is universal for $\mathfrak{g}$.

Example: $\mathfrak{k} = T^*\mathfrak{h} = \mathfrak{h}^* \rtimes \mathfrak{h}$, $\mathfrak{h}$ simple
$\kappa((\alpha, x), (\alpha', x')) = \alpha'(x) + \alpha(x')$ and
$\eta((\alpha, x), (\alpha', x')) = \alpha'(x) - \alpha(x')$ satisfy
$C(\kappa) = \mathrm{d}_{\mathfrak{k}}\eta$.

An instructive exact sequence:

Thm: (N., Wagemann; M. Bordemann '97) For each finite-dimensional Lie algebra $\mathfrak{k}$ and each trivial $\mathfrak{k}$-module V, there exists an exact sequence

$$\{0\} \to H^2(\mathfrak{k}, V) \longrightarrow H^1(\mathfrak{k}, \mathrm{Hom}(\mathfrak{k}, V)) \longrightarrow$$

$$\mathrm{Sym}^2(\mathfrak{k}, V)^{\mathfrak{k}} \xrightarrow{\quad C \quad} H^3(\mathfrak{k}, V) \longrightarrow H^2(\mathfrak{k}, \mathrm{Hom}(\mathfrak{k}, V))$$

$$\longrightarrow H^1(\mathfrak{k}, \mathrm{Sym}^2(\mathfrak{k}, V)).$$

Note: $C(\kappa) = \mathsf{d}_{\mathfrak{k}}\eta$ is equivalent to

$$\zeta(x)(y) := \kappa(x, y) - \eta(x, y) \in Z^1(\mathfrak{k}, \mathrm{Hom}(\mathfrak{k}, V)).$$

4. Classification of twists

The spaces

$$\mathfrak{z} = \overline{\Omega}^1(M,\mathbb{R}), C^\infty(M,\mathbb{R}), \Omega^1(M,\mathbb{R})$$

are $\mathcal{V}(M)$-modules and the fund. cocycles $\omega = \omega_\kappa, \omega_\eta, \omega_{\kappa,\eta}$ are $\mathcal{V}(M)$-invariant.

We thus obtain abelian extensions

$$(\mathfrak{z} \oplus_\omega C^\infty(M,\mathfrak{k})) \rtimes \mathcal{V}(M)$$

of $C^\infty(M,\mathfrak{k}) \rtimes \mathcal{V}(M)$.

Its twists are classified by the elements of the spaces
$H^2(\mathcal{V}(M), \overline{\Omega}^1(M,\mathbb{R}))$,
$H^2(\mathcal{V}(M), C^\infty(M,\mathbb{R}))$, and
$H^2(\mathcal{V}(M), \Omega^1(M,\mathbb{R}))$.

Cocycles from differential forms

Source 1: Each closed p-form $\omega \in \Omega^p(M,\mathbb{R})$ defines a Lie algebra cocycle in $Z^p(\mathcal{V}(M), C^\infty(M,\mathbb{R}))$.

Thm. (Shiga/Tsujishita '77) The kernel of the natural homomorphism

$$\Phi\colon H^\bullet_{\mathrm{dR}}(M,\mathbb{R}) \to H^\bullet(\mathcal{V}(M), C^\infty(M,\mathbb{R}))$$

is the ideal of all classes subordinated to the Pontrjagin classes $p_1, \ldots, p_{[n/4]}$ of M.

Source 2: Each closed $p+q$-form $\omega \in \Omega^{p+q}(M,\mathbb{R})$ also defines a Lie algebra p-cocycle on $\mathcal{V}(M)$ with values in

$$\overline{\Omega}^q(M,\mathbb{R}) = \Omega^q(M,\mathbb{R})/\mathrm{d}\Omega^{q-1}(M,\mathbb{R}),$$

$$\omega^{[p]}(X_1,\ldots,X_p) := [i_{X_p}\cdots i_{X_1}\omega] \in \overline{\Omega}^q(M,\mathbb{R}).$$

(Hochschild/Serre; '53):

Cocycles from affine connections

∇ affine connectin on M
$\zeta(X) := \mathcal{L}_X \nabla \in \Omega^1(M, \mathrm{End}(TM))$ (1-cocycle)

$\beta(x_1, \ldots, x_k) := \sum_{\sigma \in S_k} \mathrm{tr}(x_{\sigma(1)} \cdots x_{\sigma(k)})$
invar. pol. on $\mathfrak{gl}_d(\mathbb{R})$, $d = \dim M$.

Thm.: (Koszul '74)
$\psi_k := \beta(\zeta, \ldots, \zeta) \in Z^k(\mathcal{V}(M), \Omega^k(M, \mathbb{R}))$ and
$[\psi_k]$ does not depend on ∇.

Thm.: (Tsujishita '81; Beggs '87)
M connected, paracompact smooth manifold:
$H^\bullet(\mathcal{V}(M), \Omega^\bullet(M, \mathbb{R})) \cong$
$H^\bullet(\mathcal{V}(M), C^\infty(M, \mathbb{R})) \otimes \langle \psi_i, i = 1, \ldots, d \rangle_{alg}$

Special case: M parallelizable, $\kappa \in \Omega^1(M, \mathbb{R}^d)$ trivializing 1-form. Then

$$\mathcal{L}_X \kappa = -\theta(X) \cdot \kappa,$$

defines a crossed homo.:

$$\theta \colon \mathcal{V}(M) \to C^\infty(M, \mathfrak{gl}_d(\mathbb{R}))$$

Thm.: (Billig, N., '06) M is parallelizable $\Rightarrow$

$$\overline{\psi}_k(X_1, \ldots, X_k) := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \, \mathrm{tr}(\theta(X_{\sigma(1)}) \wedge \mathtt{d}\theta(X_{\sigma(2)}) \wedge \cdots \wedge \mathtt{d}\theta(X_{\sigma(k)}))$$

defines an $\overline{\Omega}^{k-1}(M, \mathbb{R})$-valued cocycle with

$$\mathtt{d} \circ \overline{\psi}_k = \psi_k$$

and $\psi_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \, \mathrm{tr}(\mathtt{d}\theta(X_{\sigma(1)}) \wedge \cdots \wedge \mathtt{d}\theta(X_{\sigma(k)}))$ describes Koszul's cocycles in terms of θ.

The classification of twists

Classical case: $M = \mathbb{S}^1$. Then $H^2(\mathcal{V}(\mathbb{S}^1), C^\infty(\mathbb{S}^1, \mathbb{R}))$ is 2-dimensional and

$$H^2(\mathcal{V}(\mathbb{S}^1), \mathbb{R}) = H^2(\mathcal{V}(\mathbb{S}^1), \overline{\Omega}^1(\mathbb{S}^1, \mathbb{R})) \cong \mathbb{R}$$

(Virasoro cocycle)

The following theorem is based on results of Gelfand-Fuks, Haefliger and Tsujishita.

Thm. (Y. Billig, N., '06) Form M cpt, TM trivial, $d = \dim M > 1$ we have:

• $H^2(\mathcal{V}(M), C^\infty(M, \mathbb{R})) \cong H^2_{\mathrm{dR}}(M, \mathbb{R}) \oplus H^1_{\mathrm{dR}}(M, \mathbb{R})$

Here $[\alpha] \in H^1(M, \mathbb{R})$ corresponds to $\alpha \wedge \overline{\psi}_1$, where $\overline{\psi}_1(X) = \mathrm{div} X$ if M is orientable.

• $H^2(\mathcal{V}(M), \Omega^1(M, \mathbb{R})) = \mathbb{R}[\overline{\psi}_1 \wedge \psi_1] \oplus H^1(M, \mathbb{R})$. Here $[\alpha] \in H^1(M, \mathbb{R})$ corresponds to $\alpha \wedge \psi_1$.

• $H^2(\mathcal{V}(M), \overline{\Omega}^1(M, \mathbb{R})) \cong$
$$H^3_{\mathrm{dR}}(M, \mathbb{R}) \oplus \mathbb{R}[\overline{\psi}_1 \wedge \psi_1] \oplus \mathbb{R}[\overline{\psi}_2].$$

5. Integrability to Lie group extensions

K a 1-connected Lie group, $\mathbf{L}(K) = \mathfrak{k}$.

Thm.: (Maier, N., '03) For $\kappa \in \mathrm{Sym}^2(\mathfrak{k}, V)^{\mathfrak{k}}$ the Lie algebra $\overline{\Omega}^1(M, \mathbb{R}) \oplus_{\omega_\kappa} C^\infty(M, \mathfrak{k})$ is integrable if and only if the closed left invariant 3-form $C(\kappa)^l$ on K satisfies:

$$\Pi_\kappa := \int_{\pi_3(K)} C(\kappa)^l \subseteq V \qquad \text{is discrete.}$$

This is true if $V = V(\mathfrak{k})$ and κ is universal.

Thm.: The Lie algebra extensions of $C^\infty(M, \mathfrak{k})$ by $C^\infty(M, V)$, resp., $\Omega^1(M, V)$ corresponding to cocycles of the types ω_η and $\omega_{\kappa,\eta}$ are always integrable.

Ex. If $\widehat{K}$ is a Lie group with Lie algebra $\widehat{\mathfrak{k}} = V \oplus_\eta \mathfrak{k}$, then the Lie algebra of $C^\infty(M, \widehat{\mathfrak{k}})$ is $C^\infty(M, V) \oplus_{\omega_\eta} C^\infty(M, \mathfrak{k})$.

Integrability of the twists

For the 2-cocycles on $\mathcal{V}(M)$ we have the following **sufficient** conditions for integrability of the corresponding abelian Lie algebra extension:

values in $\overline{\Omega}^1(M,\mathbb{R})$:

- $\omega \in Z^3_{\mathrm{dR}}(M,\mathbb{R})$ and $\int_{H_3(M)} \omega \subseteq \mathbb{R}$ discrete.
- $\overline{\psi}_1 \wedge \psi_1$ and $\overline{\psi}_2$ are integrable.

values in $\Omega^1(M,\mathbb{R})$:

- $\overline{\psi}_1 \wedge \psi_1$ is integrable.
- $\alpha \in Z^1_{\mathrm{dR}}(M,\mathbb{R})$ and $\int_{\pi_1(M)} \alpha \subseteq \mathbb{R}$ discrete
  $\Rightarrow \alpha \wedge \psi_1$ is integrable.

values in $C^\infty(M,\mathbb{R})$:

- $\alpha \in Z^2_{\mathrm{dR}}(M,\mathbb{R})$ with $\int_{\pi_2(M)} \alpha \subseteq \mathbb{R}$ discrete,
  $\Rightarrow \alpha^{[2]}$ is integrable (prequantization).
- $\beta \in Z^1_{\mathrm{dR}}(M,\mathbb{R})$ with $\int_{\pi_1(M)} \beta \subseteq \mathbb{R}$ discrete,
  $\Rightarrow \beta \wedge \overline{\psi}_1$ is integrable.

6. Generalizations to non-trivial bundles Cocycles of type (I)

$q\colon P \to M$ principal K-bundle, $\mathbf{L}(K) = \mathfrak{k}$

$\mathfrak{gau}(P) \cong \{f \in C^\infty(P, \mathfrak{k})\colon f(g.k) = \mathrm{Ad}(k)^{-1}.f(p)\}$ its gauge algebra.

V a K-module on which $\mathfrak{k}$ acts trivially
$\mathbb{V}$ assoc. flat vector bundle with fiber V.

For $\kappa \in \mathrm{Sym}^2(\mathfrak{k}, V)^K$ and ∇ conn. on P we obtain a 2-cocycle on $\mathfrak{gau}(P)$ by:

$$\omega_\kappa^\nabla(f, g) := [\kappa(f, \nabla g)] \in \Omega^1(M, \mathbb{V}).$$

Note: $[\omega_\kappa^\nabla]$ does not depend on ∇.

Thm. (N., Wockel, '07) If $\pi_0(K)$ is finite, then t.f.a.e.:
(1) ω_κ^∇ is integrable for each K-bundle P.

(2) ω_κ^∇ is integrable for $P = \mathbb{S}^1 \times K$.

(3) The period group $\Pi_\kappa = \int_{\pi_3(K)} C(\kappa)^l \subseteq V$ is discrete.

Cocycles of type (II)

Let $\eta \in Z^2(\mathfrak{k}, V)$ and $\widehat{\mathfrak{k}} := V \oplus_\eta \mathfrak{k}$ and

$$\mathbf{1} \to Z \to \widehat{K} \to K \to \mathbf{1}$$

a central Lie group extension with $\mathbf{L}(\widehat{K}) = \widehat{\mathfrak{k}}$.

Then the conjugation action of K on itself lifts to an action on $\widehat{K}$ and we obtain an associated Lie group bundle $\widehat{\mathcal{K}} = (P \times \widehat{K})/K$. Then we have a central Lie group extension

$$\mathbf{1} \to C^\infty(M, Z) \to \Gamma\widehat{\mathcal{K}} \to \mathsf{Gau}(P) \to \mathbf{1}$$

integrating the Lie algebra extension

$$\mathbf{0} \to C^\infty(M, V) \to \Gamma\widehat{\mathsf{Ad}}(P) \to \mathfrak{gau}(P) \to \mathbf{1}.$$

Cocycles of type (III)

Consider a cocycle

$$\omega_\kappa^\nabla \in Z^2(\mathfrak{gau}(P), \overline{\Omega}^1(M, \mathbb{V}))$$

of type (I) and assume that K acts trivially on V.

Lemma There exists a bundle map $\eta \in \mathsf{Alt}^2(\mathsf{Ad}(P), \mathbb{V})$ for which

$$\omega_{\kappa,\eta}^\nabla(f, g) := \kappa(f, \nabla g) - \kappa(g, \nabla f) - \mathtt{d}(\eta(f, g))$$

is a cocycle with values in $\Omega^1(M, V)$ if and only if $[C(\kappa)] = 0$ in $H^3(\mathfrak{k}, V)$.

Then $\omega_{\kappa,\eta}^\nabla$ is a lift of $2\omega_\kappa^\nabla$ to $\Omega^1(M, V)$.

Problems:

- Classification of central extensions for $\mathfrak{gau}(P)$.
- Integrability of the cocycles $\omega_{\kappa,\eta}^\nabla$.
- Extendability of cocycles to $\mathfrak{aut}(P) = \mathcal{V}(P)^K$.