L_∞-algebras - an overview

Two ways of making an algebraic structure (such as monoid) more flexible:

1. Categorification (Dima's talk)
2. Homotopyfication

Unclear if 1. & 2. related. Philosophically one could expect an ω-category of spaces & higher homotopies s.t.

Categorification in this ω-category $\Longleftrightarrow$ Homotopyfication

but unclear what to do.

Idea of homotopyfication:

consider axioms up to homotopy,
look for coherence constraints,
consider them vanishing up to homotopy,
⋮

can be made precise using operads and their resolutions - maybe mention later.

x x x x x

<u>1st instance</u> of homotopyfication - Stasheff ~71

Monoid: $m: X\times X \longrightarrow X$

& strict associativity $\quad m(m\times 1\!\!1_X) = m(1\!\!1_X\times m)$

Homotopyfication of monoid = A_∞ - space (= sha sp.):

·) $m_2: X\times X \longrightarrow X$

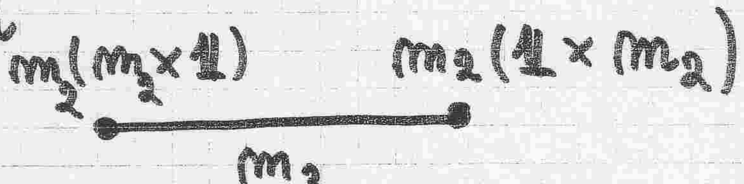
·) m_2 homotopy associative:

$\exists\ m_3: X\times X\times X\times I \longrightarrow X$

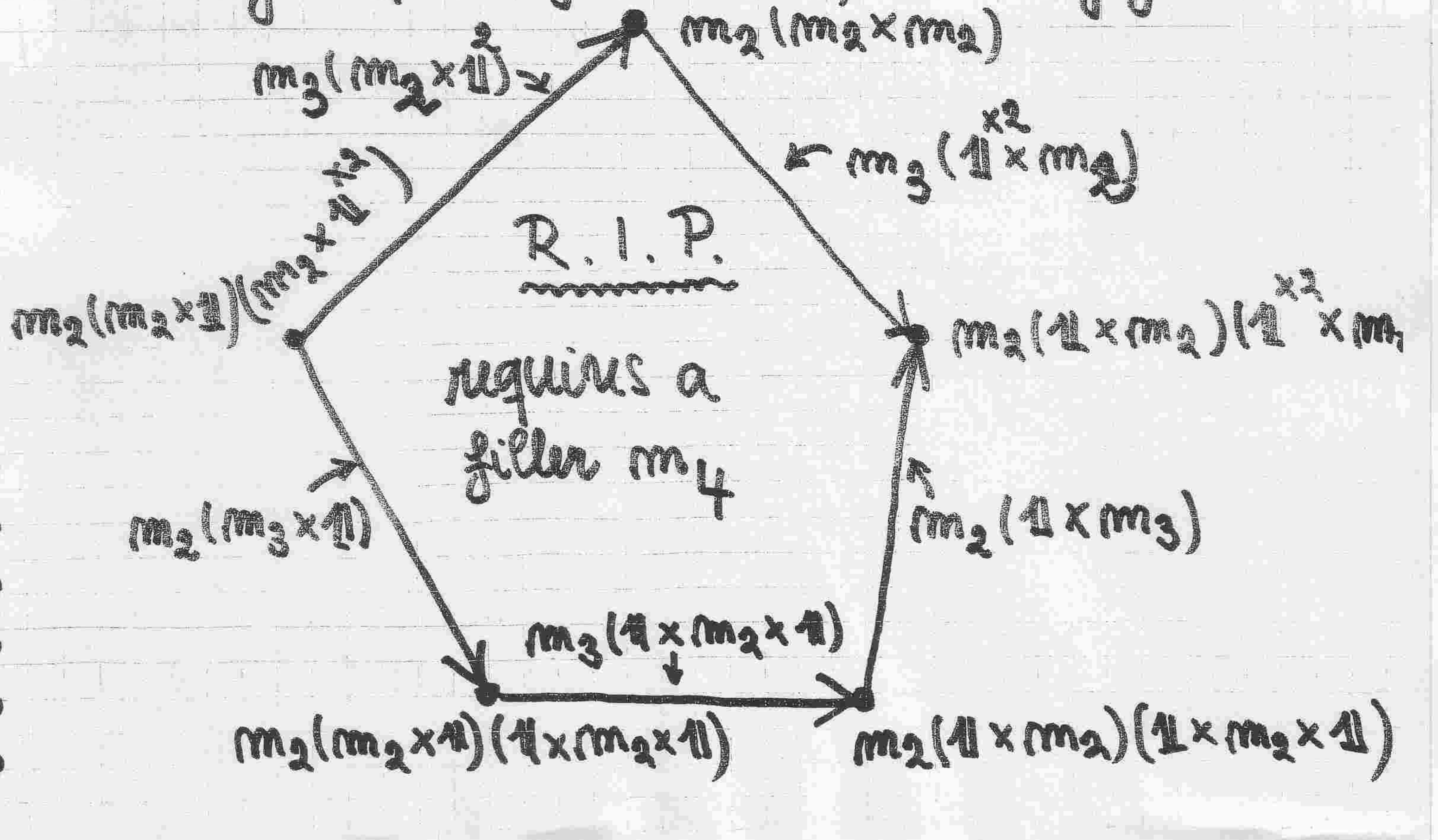
which is a homotopy between $m_2(m_2\times 1\!\!1_X)$

and $m_2(1\!\!1_X\times m_2)$

In shorthand



·) having m_2 & m_3 as above, one may form



etc.

<u>A_∞-space</u>: X & $m_k : X^{\times k} \times K_k \longrightarrow X$, $k \geq 2$,
$K_2 = Pt$, $K_3 = \bullet\!\!-\!\!\bullet$, $K_4 =$ , $\cdots$ & something satisf.

<u>Mark</u>: A_∞ space has ∞ many structure operations $m_2, m_3, m_4, \ldots$

A_∞-space .... homo....om in Top
$\exists$ also homo...om in Chain called
A_∞-algebra (aka s.h. associative algebra).
given by g.v.s. V & degree $(k-2)$-maps, $k \geq 1$,
$\mu_k : V^{\otimes k} \longrightarrow V$ that satisfy something.

———————— × × × × × ————————

L_∞-algebras (aka s.h. Lie) are homotopyfication of Lie algebras. Since not given by diagrams of set maps, only linear = chain version exists.

Since L_∞ algebras central objects, I give definition w/ all gory details.

$L \cdots$ g.v.s., ΛL = free graded comm. ass. algebra
$$\cong \mathbb{k}[V^{even}] \otimes E(V^{odd})$$

$x_1, \dots, x_k \in L$ & $\sigma \in \Sigma_k$ define $\varepsilon(\sigma)$ by

$$x_1 \wedge \dots \wedge x_k = \varepsilon(\sigma)\, x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(k)}$$

$\varepsilon(\sigma) = \varepsilon(\sigma; x_1, \dots, x_k) \in \{-1, 1\}$ called Koszul sign.

Define also $\underline{\chi(\sigma) := \mathrm{sgn}(\sigma)\,\varepsilon(\sigma)}$.

One says $\sigma \in \Sigma_k$ is $\underline{(i, m-i)\text{-unshuffle}}$, $1 \leq i \leq m$, iff $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(m)$.

Definition L_∞ is $(L, \ell_1, \ell_2, \ell_3, \dots)$ st.

$$\ell_k : \bigotimes^k L \longrightarrow L \text{ is a degree } k-2 \text{ map}$$

s.t.

- ℓ_k is graded antisymmetric:

$$\ell_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma)\, \ell_k(x_1, \dots, x_k)$$

- $\forall\, m \geq 1$, (L_m) sat.:

$$\sum_{i+j=m+1} \ \sum_{\sigma \in (i,m-i)\text{ unshuffles}} \chi(\sigma)(-1)^{i(j-1)} \, \ell_j\big(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}\big) = 0$$

Ex. $\ell_1 : L \longrightarrow L$ degree -1, (L_1) means $\ell_1^2 = 0$

$\Rightarrow (L, \ell_1)$ is a chain complex.

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•) $l_2 = [\ ,\ ] : L \otimes L \longrightarrow L$ antisymmetric

& L_2 says:

$$l_1([x,y]) = [l_1(x), y] + (-1)^{|x|}[x, l_1(y)]$$

$$\Updownarrow$$

$[\ ,\ ] : L \otimes L \longrightarrow L$ is a chain map.

•) $l_3 : L^{\otimes 3} \longrightarrow L$ is a degree $+1$ antisymmetric map s.t. L_3 (in what follows I ignore signs)

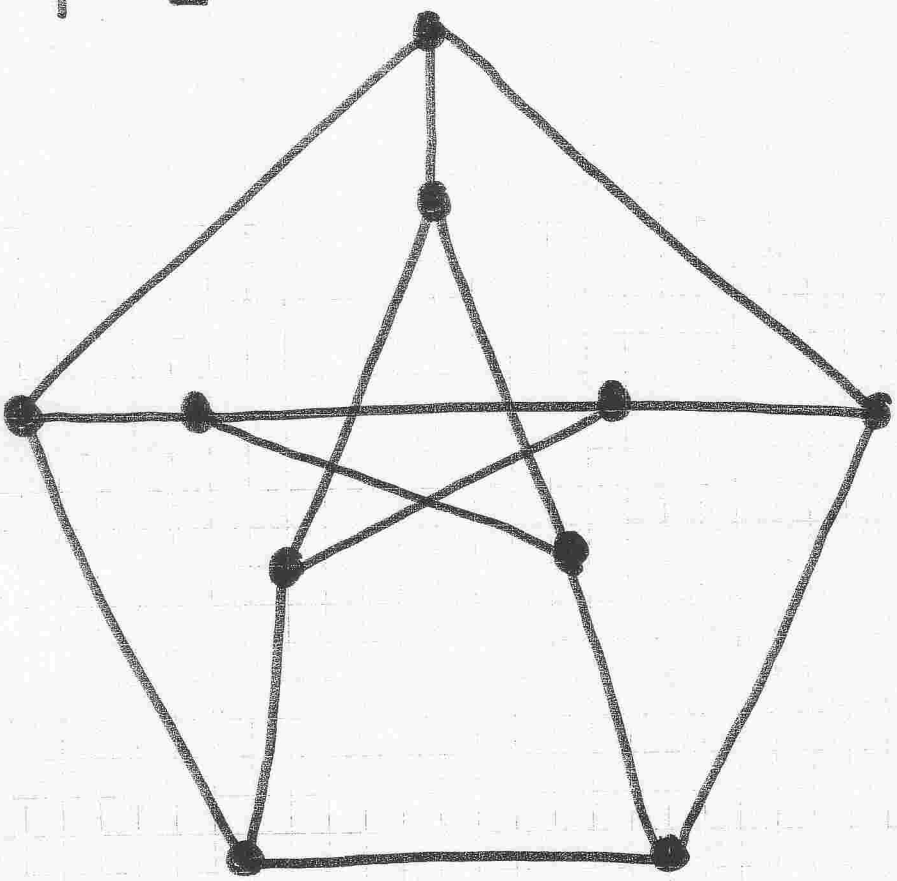
$$(-1)^{|x||y|}[[x,y],z] + (-1)^{|y||z|}[[z,x],y] + (-1)^{|x||y|}[[y,z],x]$$
$$= \pm l_1 l_3(x,y,z) \pm l_3(l_1(x),y,z) \pm l_3(x,l_1(y),z) \pm l_3(x,y,l_1(z))$$

The meaning is that Jacoby$([\ ,\ ])$ considered as a map $L^{\otimes 3} \longrightarrow L$ is chain homotopic to zero up to an explicit homotopy l_3, e.t.c.

Examples:
- ·) dg Lie algebras ($l_3 = l_4 = \cdots 0$)
- ·) BRST complex of matter & ghosts of closed string FT
- will try to make comments $\longrightarrow$ ·) deformation complex ("Deligne groupoid") of an algebraic str.
- ·) $\pi_*(\Omega S) \otimes \mathbb{Q}$ w homotopy Massey products (= higher Whitehead)

No polyhedra, but still some graphs.

Relax a while enjoying positive vibrations steaming from the following L_∞ analog of K_4 = ⬠



L_∞ algebras form a <u>category</u>:

$F : (L', \ell'_1, \ell'_2, \dots) \longrightarrow (L'', \ell''_1, \ell''_2, \dots)$

given as $F = (f_1, f_2, f_3, \dots)$, $f_i : \otimes^i L' \to L''$

f_i degree $i-1$ s.t.

- $f_1 : (L', \ell'_1) \longrightarrow (L'', \ell''_1)$ is a chain map,
- $f_2 : L' \otimes L' \to L''$ measures deviation from f_1 being a $(L', [-,-]') - (L'', [-,-]'')$ homomorphism, ...

F called <u>strict</u> if $f_2 = f_3 = \dots = 0$

$\Rightarrow$ f_1 commutes with everything

F called <u>weak equivalence</u> iff

$$H_*(f_1) : H_*(L', \ell'_1) \longrightarrow H_*(L'', \ell''_1)$$

is iso.

a <u>compact</u> definition of L_∞-algebras

$(L, \ell_1, \ell_2, \dots)$ as before, denote $\downarrow L = L[1]$.

$\forall\, \ell_k$ induces a degree -1 map

$$\delta_k : \Lambda^k \downarrow L \longrightarrow \downarrow L$$

System $\{\delta_k\}_{k \geq 1}$ uniquely extends into a coderivation

$$\delta \in \mathrm{CoDer}^{-1}(\Lambda^c \downarrow L)$$

Axioms $L_1, L_2, \dots \iff \delta^2 = 0$!

<u>Theorem</u>: 1-1 correspondence between

·) L_∞ structures on a g.v.s. L

·) coder. as above $\overline{w}$ $\delta^2 = 0$.

Therefore one may <u>define</u> L_∞ algebras as coderivations $\overline{w}$ $\delta^2 = 0$.

This definition is compact but the idea is obscured. Maps in this setting are d.g. coalgebra homomorphisms

$$F : (\Lambda^c \downarrow L', \delta') \longrightarrow (\Lambda^c \downarrow L'', \delta'').$$

The best Definition:

Lie algebras are algebras over an operad $\mathcal{L}ie$.

$\mathcal{L}ie$ has unique minimal model $\mathcal{L}_\infty$.

Definition: L_∞ algebras are algebras over $\mathcal{L}_\infty$.

Advantage: short and conceptual: minimal model is a particular cofibrant replacement of $\mathcal{L}ie$. By a general theory this implies the following CRUTIAL TRANSFER PROPERTY which explains the importance and omnipresence of L_∞-algebras.

Suppose given

$$\begin{tikzcd} (L', \partial') \arrow[r, bend right, "f"'] & (L'', \partial'') \arrow[l, bend right, "g"'] \end{tikzcd}$$

diagram of chain complexes s.t.

$$gf \overset{h}{\sim} \mathbb{1}_{L'} \text{ (chain homotopy)}$$

so g is a left homotopy inverse of f.

Suppose moreover given L_∞ algebra $L' = (L', \partial', \ell'_2, \ell'_3, \ell'_4, \dots)$

$$\Downarrow$$

$\exists$ L_∞ structure $L'' = (L'', \partial'', \ell''_2, \ell''_3, \ell''_4, \dots)$

& Extensions of f & g into L_∞-maps

$F = (f_1, f_2, f_3, \dots)$, $G = (g_1, g_2, g_3, \dots)$

& h into $H = (h_1, h_2, \dots)$ s.t.

$$GF \overset{H}{\sim} \mathbb{1}_V \text{ in } L_\infty.$$

One may say that L'' <u>induced</u> by L'.

Corollary: $(V',\partial') \sim (V'',\partial'')$
homotopy ~~invariant~~ equivalent chain complexes $\Rightarrow$ Each L_∞ structure on (V',∂') induces an L_∞-structure on (V'',∂'').

Still more concisely: L_∞ algebras are homotopy invariant structures.

Not true for dg-Lie algebras: if (V',∂') bears a dg-Lie structure (= part. case of L_∞) $\Rightarrow (V'',\partial'')$ has an induced L_∞, not dg-Lie.

So L_∞ are indeed homotopy invariant versions of dg-Lie ("homotopy invariant envelope").

Explains origin of L_∞ in BRST.

Deformations & L_∞-algebras (leaving details out).

Traditional picture: Deformations given by a dg Lie algebra $\mathfrak{g}$:

$$MC(\mathfrak{g}) := \{ s \in \mathfrak{g}^1 ; ds + \tfrac{1}{2}[s,s] = 0 \},$$

one suitably defines $G(\mathfrak{g})$ (the gauge gp.)

& $\underline{\underline{Def}}(\mathfrak{g}) = MC(\mathfrak{g}) / G(\mathfrak{g})$

(Gerstenhaber, Grothendieck, Deligne, ..)

Can be extended to L_∞-setting:

$L = (L, \ell_1, \ell_2, \ell_3, \dots)$,

$MC(L) = \{ s \in L^1, \ell_1(s) + \frac{1}{2}\ell_2(s,s) + \frac{1}{6}\ell_3(s,s,s) + \dots = 0 \}$

$G(L)$ defined suitably.

& $Def(L) := MC(L) / G(L)$.

Reasons for this:

·) Some problems like defs. of bialgebras described by fully fledged L_∞-algebras.

·) L_∞ approach reveals the nature:

THEOREM (MK). $L \mapsto \mathcal{D}ef(L)$
extends to a functor $\mathbb{L}_\infty \longrightarrow \mathcal{S}ets$.
In particular, $\forall$ L_∞-map $F : L' \longrightarrow L''$
induces a map ~~[illegible]~~
$$\mathcal{D}ef(F) : \mathcal{D}ef(L') \longrightarrow \mathcal{D}ef(L'').$$
If F is a WE $\Rightarrow$ $\mathcal{D}ef(F)$ is iso.

Corollary $\mathfrak{g}'$ & $\mathfrak{g}''$ ordinary dg-Lie.
If $\mathfrak{g}'$ & $\mathfrak{g}''$ WE in L_∞ $\Rightarrow$ $\mathcal{D}ef(\mathfrak{g}') \cong \mathcal{D}ef(\mathfrak{g}'')$

$\uparrow$

no way to formulate classically.

Used in MK's proof of Deformation quant.

$\mathfrak{g}'$ = graded Lie alg. of polyvector fields,
$\mathcal{D}ef(\mathfrak{g}')$ = Poisson structures

$\mathfrak{g}''$ = dg Lie algebra of polydifferential operators, $\mathcal{D}ef(\mathfrak{g}'') = *$-products

$\mathfrak{g}', \mathfrak{g}''$ not isomorphic, nor anyhow related in dg Lie. But:

Formality thm: $\mathfrak{g}'$ & $\mathfrak{g}''$ WE in L_∞.

$$\Rightarrow \mathrm{Def}(\mathfrak{g}') \cong \mathrm{Def}(\mathfrak{g}'')$$

$$\Downarrow$$

$\exists$ of Def. quantization