Holomorphic Poisson Structures and Groupoids

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Outline

Holomorphic Poisson Manifolds

DEFINITION: A holomorphic Poisson manifold is a complex manifold X whose sheaf of holomorphic functions $\mathcal{O}_X$ is a sheaf of Poisson algebras.

$$f, g \in \mathcal{O}_X \quad \Longrightarrow \quad \{f, g\} \in \mathcal{O}_X$$

$$\{f, g\} = \pi(df, dg) = \sum_{i<j} \pi_{ij}\left(\partial_{z_i} f \partial_{z_j} g - \partial_{z_j} f \partial_{z_i} g\right)$$

$$\pi = \sum_{i<j} \pi_{ij} \partial_{z_i} \wedge \partial_{z_j} \qquad \text{s.t. } \pi_{ij} \in \mathcal{O}_X$$

More

FACT: $(\mathcal{O}_X, \{\cdot, \cdot\})$ is holomorphic Poisson
IFF $\pi \in \Gamma(\wedge^2 T^{1,0} X)$ satisfies $[\pi, \pi] = 0$ and $\bar{\partial}\pi = 0$

Real and imaginary parts of π

$\wedge^2 T^{1,0}X \subset \wedge^2 T_{\mathbb{C}}X = \wedge^2 TX \oplus i \wedge^2 TX$
$\pi = \pi_R + i\pi_I$ where $\pi_R, \pi_I \in \Gamma(\wedge^2 TX)$

QUESTION: Are π_R and π_I real Poisson structures?
And conversely, given two real poisson structures π_R and π_I, when does $\pi = \pi_R + i\pi_I$ define a holomorphic Poisson structure?

Poisson Nijenhuis structures

[Magri-Morosi]

Recall that a Poisson Nijenhuis structure on a manifold X consists of a pair (π, N), where π is a Poisson tensor on X and $N : TX \to TX$ is a Nijenhuis tensor such that the following compatibility conditions are satisfied:

$$N \circ \pi^\sharp = \pi^\sharp \circ N^*$$
$$[\alpha, \beta]_{\pi_N} = [N^*\alpha, \beta]_\pi + [\alpha, N^*\beta]_\pi - N^*[\alpha, \beta]_\pi$$

where π_N is the bivector field on X defined by the relation $\pi_N^\sharp = \pi^\sharp \circ N^*$ and for any bivector field $\hat{\pi}$ on M,

$$[\alpha, \beta]_{\hat{\pi}} := \mathcal{L}_{\hat{\pi}^\sharp \alpha}(\beta) - \mathcal{L}_{\hat{\pi}^\sharp \beta}(\alpha) - d\big(\hat{\pi}(\alpha, \beta)\big), \qquad \forall \alpha, \beta \in \Omega^1(M).$$

QUESTION: Are π_R and π_I real Poisson structures?

THEOREM: Given a complex manifold X with associated almost complex structure J, the following are equivalent:

1. $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0}X)$ is a holomorphic Poisson bivector field;
2. (π_I, J) is a Poisson Nijenhuis structure on X and $\pi_R^\sharp = \pi_I^\sharp \circ J^*$;
3. $\mathbb{J}_\pi = \begin{pmatrix} J & \pi_I^\sharp \\ 0 & -J^* \end{pmatrix}$ $(\in \mathrm{End}(TM \oplus T^*M))$ is a generalized complex structure and $\pi_R^\sharp = \pi_I^\sharp \circ J^*$.

Thus (π_R, π_I) is a biHamiltonian structure on X.

Symplectic Foliations

Fact: Let (X, π) be a holomorphic Poisson manifold, and π_R and π_I the real and imaginary parts of π.

Then the symplectic foliations of π_R and π_I coincide, and their leaves are exactly the holomorphic symplectic leaves of π.

Holomorphic Vector Bundles (HVB)

Given a HVB $A \xrightarrow{p} X$,
$\mathcal{A}$ denotes its **sheaf of holomorphic sections**
and $\mathcal{A}_\infty$ its **sheaf of smooth sections.**
Clearly, $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-modules
while $\mathcal{A}_\infty$ is a sheaf of $C^\infty(X)$-modules.
Moreover $\mathcal{A}$ is a subsheaf of $\mathcal{A}_\infty$.

The tangent bundle $TX \to X$ of a complex manifold X is naturally a HVB. We will denote its sheaf of holomorphic sections, i.e. the sheaf of holomorphic vector fields, by Θ_X.

The cotangent bundle $T^*X \to X$ of a complex manifold X is naturally a HVB. We will denote its sheaf of holomorphic sections, i.e. the sheaf of holomorphic 1-forms, by Ω_X.

Holomorphic Lie Algebroids (HLA)

DEFINITION: A HLA is a HVB $A \to X$, equipped with a holomorphic bundle map $A \xrightarrow{\rho} TX$, called the anchor map, and a structure of sheaf of complex Lie algebras on $\mathcal{A}$, such that

1. the anchor map ρ induces a homomorphism of sheaves of complex Lie algebras from $\mathcal{A}$ to Θ_X;
2. and the Leibniz identity

$$[V, fW] = \big(\rho(V)f\big)W + f[V, W]$$

holds for all $V, W \in \mathcal{A}(U)$, $f \in \mathcal{O}_X(U)$ and all open subsets U of X.

EXAMPLES:

1. The tangent bundle $TX \to X$ of a complex manifold X is naturally a HLA.
2. The cotangent bundle $T^*X \to X$ of a holomorphic Poisson manifold (X, π) is a HLA with anchor $\pi^\sharp : T^*X \to TX$ and bracket

$$[\alpha, \beta]_\pi = L_{\pi^\sharp \alpha}\beta - L_{\pi^\sharp \beta}\alpha + \frac{1}{2} d\big(\pi(\alpha, \beta)\big),$$

for all $\alpha, \beta \in \Omega^1_X$.

Underlying Real Lie Algebroid (1/2)

Some special real Lie algebroids give rise to HLAs.

By forgetting the complex structure, a HVB $A \to X$ becomes a real (smooth) vector bundle, and a HVB map $\rho : A \to TX$ becomes a real (smooth) vector bundle map.

Let $A \to X$ be a HVB whose underlying real vector bundle is endowed with a Lie algebroid structure $(A, \rho, [\cdot, \cdot])$ such that, for any open subset $U \subset X$, **(1)** $[\mathcal{A}(U), \mathcal{A}(U)] \subset \mathcal{A}(U)$ and **(2)** the restriction of the Lie bracket $[\cdot, \cdot]$ to $\mathcal{A}(U)$ is $\mathbb{C}$-linear. Then the restriction of $[\cdot, \cdot]$ and ρ from $\Gamma(A)$ to $\mathcal{A}$ makes A a HLA.

Underlying Real Lie Algebroid (2/2)

Actually, any HLA can be obtained out of such a real Lie algebroid, in a unique way.

FACT: Given a structure of HLA on a HVB $A \to X$ with anchor map $A \xrightarrow{\rho} TX$, there exists a unique structure of real smooth Lie algebroid on the vector bundle $A \to X$ with respect to the same anchor map ρ such that the inclusion of sheaves $\mathcal{A} \subset \mathcal{A}_\infty$ is a morphism of sheaves of real Lie algebras.

In the sequel, we will use A_R to denote the **underlying real Lie algebroid** of a HLA A.

Underlying Imaginary Lie Algebroid

1. Take a HLA $(A \to X, \rho, [\cdot,\cdot])$.
2. Consider the bundle map $j : A \to A$ defining the fiberwise complex structure on A.
3. **Fact:** The Nijenhuis torsion of j w.r.t. the bracket of the real Lie algebroid A_R vanishes.
4. Therefore, one can define a new (real) Lie algebroid structure on A, denoted by $(A \to X, \rho_j, [\cdot,\cdot]_j)$, where the anchor ρ_j is $\rho\circ j$ and the bracket on $\Gamma(A)$ is given by

   $$[V, W]_j = [jV, W] + [V, jW] - j[V, W], \qquad \forall V, W \in \Gamma(A).$$

   [Cariñena-Grabowski-Marmo]
5. $A_I := (A \to X, \rho_j, [\cdot,\cdot]_j)$ will be called the **underlying imaginary Lie algebroid**
6. $j : A_I \to A_R$ is a Lie algebroid isomorphism

Holomorphic Lie-Poisson structures

The Lie algebroid structures on a given vector bundle are in 1-1 correspondence with the so-called fiberwise linear Poisson structures on the dual bundle.

This correspondence extends to the holomorphic context.

FACT: Let $A \to X$ be a holomorphic vector bundle.
The following are equivalent:

1. A is a holomorphic Lie algebroid;
2. there exists a fiberwise-linear holomorphic Poisson structure on $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$.

More

1. Consider a holomorphic Lie algebroid $(A \to X, \rho, [\cdot, \cdot])$.
2. Its complex dual bundle $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ is a fiberwise linear holomorphic Poisson manifold, whose holomorphic Poisson tensor is denoted by π.
3. Let π_R and π_I be its real and imaginary parts. Then $\pi_{\Re} := \Psi_*^{-1}\pi_R$ and $\pi_{\Im} := \Psi_*^{-1}\pi_I$ are fiberwise $\mathbb{R}$-linear Poisson tensors on the real dual bundle $\mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R})$. ▸ More
4. These Poisson structures therefore correspond to real Lie algebroids on $A \to X$, which are denoted by $(A \to X, \rho_{\Re}, [\cdot, \cdot]_{\Re})$ and $(A \to X, \rho_{\Im}, [\cdot, \cdot]_{\Im})$, respectively.

QUESTION: Obtain an explicit description of the Lie algebroid structures $A_{\Re}$ and $A_{\Im}$ in terms of the holomorphic Lie algebroid structure on A.

QUESTION: Obtain an explicit description of the Lie algebroid structures $A_{\Re}$ and $A_{\Im}$ in terms of the holomorphic Lie algebroid structure on A.

FACT: Let $(A \to X, \rho, [\cdot,\cdot])$ be a holomorphic Lie algebroid.

1. The Lie algebroid $(A \to X, 4\rho_{\Re}, 4[\cdot,\cdot]_{\Re})$ is isomorphic to the real Lie algebroid A_R;
2. The Lie algebroid $(A \to X, -4\rho_{\Im}, -4[\cdot,\cdot]_{\Im})$ is isomorphic to the imaginary Lie algebroid A_I.

Equivalent definition of HLA

Let $(A, \rho, [\cdot, \cdot])$ be a real Lie algebroid, where $A \to X$ is a holomorphic vector bundle. The following are equivalent:

1. $(A, \rho, [\cdot, \cdot])$ is a holomorphic Lie algebroid;
2. if J_A and J_X denote the almost complex structures on A and X respectively, the map

$$\begin{array}{ccc} TA & \xrightarrow{J_A} & TA \\ \downarrow & & \downarrow \\ TX & \xrightarrow[J_X]{} & TX \end{array}$$

   defines a Lie algebroid isomorphism.

Integration

A (holomorphic) Lie algebroid is integrable if there exists an s-connected and s-simply connected (holomorphic) Lie groupoid of which it is the infinitesimal version.

QUESTION: Given a holomorphic Lie algebroid A with underlying real Lie algebroid A_R, what is the relation between the integrability of A and the integrability of A_R?

THEOREM: A_I is integrable
IFF A is integrable
IFF A_R is integrable

Holomorphic Symplectic Groupoids/Realizations

A **holomorphic symplectic groupoid** is a holomorphic Lie groupoid $\Gamma \rightrightarrows X$ together with a holomorphic symplectic 2-form $\omega \in \Omega^{2,0}(\Gamma)$ such that the graph of multiplication $\Lambda \subset \Gamma \times \Gamma \times \bar{\Gamma}$ is a Lagrangian submanifold, where $\bar{\Gamma}$ stands for the Γ equipped with the opposite symplectic structure.

Given a holomorphic symplectic groupoid $\Gamma \rightrightarrows X$, its holomorphic Lie algebroid is isomorphic to the cotangent Lie algebroid $(T^*X)_\pi \to X$, where π is the induced holomorphic Poisson structure on X.

Conversely, a holomorphic Poisson manifold (X, π) is said to be *integrable* if it is the induced holomorphic Poisson structure on the unit space of a holomorphic symplectic groupoid $\Gamma \rightrightarrows X$. We say that $\Gamma \rightrightarrows X$ *integrates* the holomorphic Poisson structure (X, π).

THEOREM: A holomorphic Poisson manifold is integrable if, and only if, either its real or its imaginary part is integrable as a real Poisson manifold.

This theorem can be derived from the equivalence between holomorphic Poisson manifolds and Poisson Nijenhuis structures. See Crainic or S-Xu.

Holomorphic Lie Algebroid Cohomology

1. Let $A \to X$ be a HLA.
2. We have got the complex of sheaves over X

$$\Omega_A^\bullet : \ \Omega_A^0 \xrightarrow{d_A} \Omega_A^1 \xrightarrow{d_A} \cdots \xrightarrow{d_A} \Omega_A^k \xrightarrow{d_A} \Omega_A^{k+1} \xrightarrow{d_A} \cdots$$

   where Ω_A^k stands for the sheaf of *holomorphic* sections of $\wedge^k A^* \to X$ (and $\Omega_A^0 = \mathcal{O}_X$).
3. By definition, the *holomorphic* Lie algebroid cohomology of A is the cohomology $H^*(X, \Omega_A^\bullet)$ of this complex of sheaves.

EXAMPLE: X = complex mfd, $A = TX$

$$H^*(X, \Omega_X^\bullet) \simeq H^*_{\mathrm{DR}}(X)$$

QUESTION: Given an arbitrary *holomorphic* Lie algebroid A, find a *complex* Lie algebroid L whose cohomology groups are isomorphic to those of A.

ANSWER: $L = T^{0,1}X \bowtie A^{1,0}$

When $A = (T^*X)_\pi$ (where X is a holomorphic Poisson manifold), then L is the $\sqrt{-1}$-eigenbundle of the generalized complex structure $\mathbb{J}_{4\pi}$.

Matched Pair of Lie Algebroids

[Lu,Mackenzie,Mokri]

A and B are ($\mathbb{C}$ or $\mathbb{R}$) Lie algebroids over same base mfd M

B is an A-module: $\Gamma(A) \otimes \Gamma(B) \to \Gamma(B) : (X, Y) \mapsto \nabla_X Y$

A is a B-module: $\Gamma(B) \otimes \Gamma(A) \to \Gamma(A) : (Y, X) \mapsto \nabla_Y X$

Compatibility conditions:

$$[a(X), b(Y)] = -a(\nabla_Y X) + b(\nabla_X Y),$$
$$\nabla_X[Y_1, Y_2] = [\nabla_X Y_1, Y_2] + [Y_1, \nabla_X Y_2] + \nabla_{\nabla_{Y_2} X} Y_1 - \nabla_{\nabla_{Y_1} X} Y_2,$$
$$\nabla_Y[X_1, X_2] = [\nabla_Y X_1, X_2] + [X_1, \nabla_Y X_2] + \nabla_{\nabla_{X_2} Y} X_1 - \nabla_{\nabla_{X_1} Y} X_2,$$

where $X_1, X_2, X \in \Gamma(A)$ and $Y_1, Y_2, Y \in \Gamma(B)$.

a = anchor of A
b = anchor of B

Given a matched pair (A, B) of Lie algebroids, there is a Lie algebroid structure $A \bowtie B$ on the direct sum vector bundle $A \oplus B$, with anchor $c(X \oplus Y) = a(X) + b(Y)$ and bracket

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = \big([X_1, X_2] + \nabla_{Y_1} X_2 - \nabla_{Y_2} X_1\big) \oplus \big([Y_1, Y_2] + \nabla_{X_1} Y_2 - \nabla_{X_2} Y_1\big).$$

Conversely, if $A \oplus B$ has a Lie algebroid structure for which $A \oplus 0$ and $0 \oplus B$ are Lie subalgebroids, then the representations ∇ defined by

$$[X \oplus 0, 0 \oplus Y] = -\nabla_Y X \oplus \nabla_X Y$$

endow the couple (A, B) with a matched pair structure.

EXAMPLE:

X = complex mfd

Set $\nabla_{X^{0,1}} X^{1,0} = \mathrm{pr}^{1,0}[X^{0,1}, X^{1,0}]$
and $\nabla_{X^{1,0}} X^{0,1} = \mathrm{pr}^{0,1}[X^{1,0}, X^{0,1}]$
for all $X^{0,1} \in \mathfrak{X}^{0,1}(X)$ and $X^{1,0} \in \mathfrak{X}^{1,0}(X)$.

Then $(T^{0,1}X, T^{1,0}X)$ is a matched pair.

$T^{0,1}X \bowtie T^{1,0}X \simeq T_{\mathbb{C}}X$ as CLAs

More generally, given a holomorphic Lie algebroid A, the couple $(A^{0,1}, A^{1,0})$ is a matched pair of Lie algebroids and $A^{0,1} \bowtie A^{1,0}$ is isomorphic, as a complex Lie algebroid, to $A_{\mathbb{C}}$.

WELL-KNOWN FACT: Let E be a complex vector bundle over a complex manifold X. Then E is a holomorphic vector bundle if, and only if, E is a $T^{0,1}X$-module — i.e. there exists a *flat* $T^{0,1}X$-connection on E.

FACT: Let A be a holomorphic Lie algebroid over a complex manifold X. Then the couple $(T^{0,1}X, A^{1,0})$ is naturally a matched pair of complex Lie algebroids. ▸ More

Conversely, given a complex manifold X and a matched pair $(T^{0,1}X, B)$, where B is a complex Lie algebroid over X whose anchor takes its values in $T^{1,0}X$, there exists a holomorphic Lie algebroid A such that $B \simeq A^{1,0}$ as complex Lie algebroids.

THEOREM: For any HLA $A \to X$,

$$\boxed{H^*(X, \Omega_A^\bullet) = H^*(T^{0,1}X \bowtie A^{1,0}, \mathbb{C})}.$$

IDEA OF THE PROOF: Use a double complex

- whose total cohomology is $H^*(T^{0,1}X \bowtie A^{1,0}, \mathbb{C})$
- and which is a resolution of the complex of sheaves $\Omega_A^\bullet$.

FACT: Let A and B be a pair of Lie algebroids over M with mutual actions ∇. The couple (A, B) is a matched pair **IFF** the diagram

$$\begin{array}{ccc} \Gamma(\wedge^k A^* \otimes \wedge^l B^*) & \xrightarrow{\partial_A} & \Gamma(\wedge^{k+1} A^* \otimes \wedge^l B^*) \\ \Big\downarrow{\scriptstyle \partial_B} & & \Big\downarrow{\scriptstyle \partial_B} \\ \Gamma(\wedge^k A^* \otimes \wedge^{l+1} B^*) & \xrightarrow[\partial_A]{} & \Gamma(\wedge^{k+1} A^* \otimes \wedge^{l+1} B^*) \end{array}$$

commutes, where ∂_A and ∂_B denote the Lie algebroid cohomology differential operators of A with values in the module $\wedge^\bullet B^*$ and of B with values in the module $\wedge^\bullet A^*$, respectively.

More

The Lie algebroid cohomology of $A \bowtie B$ (with trivial coefficients) is isomorphic to the total cohomology of this double complex.

Taking $A = T^{0,1}X$ and $B = A^{1,0}$ in this double complex, by the holomorphic Poincaré Lemma, we obtain a resolution of the complex of sheaves $\Omega_A^\bullet$:

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow{\scriptstyle d_A} & & \uparrow{\scriptstyle d_A^{1,0}} & & \uparrow{\scriptstyle d_A^{1,0}} & & \\
0 & \longrightarrow & \Omega_A^2 & \xrightarrow{\bar\partial} & \Omega_X^{0,0} \otimes_{C_X^\infty} \mathcal{A}_\infty^{2,0} & \xrightarrow{\bar\partial} & \Omega_X^{0,1} \otimes_{C_X^\infty} \mathcal{A}_\infty^{2,0} & \xrightarrow{\bar\partial} & \cdots \\
 & & \uparrow{\scriptstyle d_A} & & \uparrow{\scriptstyle d_A^{1,0}} & & \uparrow{\scriptstyle d_A^{1,0}} & & \\
0 & \longrightarrow & \Omega_A^1 & \xrightarrow{\bar\partial} & \Omega_X^{0,0} \otimes_{C_X^\infty} \mathcal{A}_\infty^{1,0} & \xrightarrow{\bar\partial} & \Omega_X^{0,1} \otimes_{C_X^\infty} \mathcal{A}_\infty^{1,0} & \xrightarrow{\bar\partial} & \cdots \\
 & & \uparrow{\scriptstyle d_A} & & \uparrow{\scriptstyle d_A^{1,0}} & & \uparrow{\scriptstyle d_A^{1,0}} & & \\
0 & \longrightarrow & \Omega_A^0 & \xrightarrow{\bar\partial} & \Omega_X^{0,0} \otimes_{C_X^\infty} \mathcal{A}_\infty^{0,0} & \xrightarrow{\bar\partial} & \Omega_X^{0,1} \otimes_{C_X^\infty} \mathcal{A}_\infty^{0,0} & \xrightarrow{\bar\partial} & \cdots
\end{array}
$$

where $\Omega_X^{0,k} \otimes_{C_X^\infty} \mathcal{A}_\infty^{l,0}$ denotes the sheaf of sections of the complex vector bundle $(T^{0,k}X)^* \otimes \wedge^l A^{1,0} \to X$.

THEOREM: Let (X,π) be a holomorphic Poisson manifold. The following cohomologies are all isomorphic:

1. the **holomorphic Poisson cohomology** of (X,π);
2. the holomorphic Lie algebroid cohomology of $(T^*X)_\pi$;
3. the complex Lie algebroid cohomology of $T_X^{0,1} \bowtie (T^{1,0}X)^*_\pi$;
4. the total cohomology of the double complex

$$\begin{array}{ccccc}
\cdots & & \cdots & & \\
\uparrow{\scriptstyle d_\pi} & & \uparrow{\scriptstyle d_\pi} & & \\
\Omega^{0,k}(X, T^{l+1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,k+1}(X, T^{l+1,0}X) & \xrightarrow{\bar\partial} & \cdots \\
\uparrow{\scriptstyle d_\pi} & & \uparrow{\scriptstyle d_\pi} & & \\
\Omega^{0,k}(X, T^{l,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,k+1}(X, T^{l,0}X) & \xrightarrow{\bar\partial} & \cdots
\end{array}$$

More

In particular, if $\pi = 0$, we obtain:

$$H^k\big((T^*X)_0\big) = \bigoplus_{i+j=k} H^i(X, T^{j,0}).$$

Since the restrictions of the operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial x}$ and $-i\frac{\partial}{\partial y}$ from $C^\infty(X, \mathbb{C})$ to $\mathcal{O}_X$ are one and the same, **there are different natural ways one can extend a differential operator defined on $\mathcal{O}_X$ to an operator defined on $C^\infty(X, \mathbb{C})$.**

Here $\pi = \sum \pi_{ij}\partial_{z_i} \wedge \partial_{z_j} \in \Gamma(\wedge^2 T^{1,0}X)$ is a bidifferential operator on $C^\infty(X.\mathbb{C})$. **We have already made the choice** of the extension.

Go back

Note that the complex dual $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ of a holomorphic vector bundle $A \to X$ is again a holomorphic manifold, which is also a holomorphic vector bundle over X. We denote by $p : \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \to X$ the projection onto the base manifold. There is a one-one correspondence between holomorphic sections $V \in \mathcal{A}(U)$ and fiberwise-linear holomorphic functions l_V on $\mathrm{Hom}_{\mathbb{C}}(A|_U, \mathbb{C})$: $\forall \alpha \in \mathrm{Hom}_{\mathbb{C}}(A|_U, \mathbb{C})$

$$l_V(\alpha) = \alpha(V|_{p(\alpha)}).$$

Here the Lie algebroid structure on $(A, \rho, [\cdot, \cdot])$ and the Poisson structure on $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ are related by the following equations:

$$\begin{aligned} \{p^* f, l_V\} &= p^* \big(\rho(V)(f)\big) \\ \{l_V, l_W\} &= l_{[V,W]} \end{aligned}$$

for any $V, W \in \mathcal{A}(U)$ and $f \in \mathcal{O}_X(U)$.

Go back

Given a complex vector bundle $A \to X$, we denote its complex and real dual bundles by $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \to X$ and $\mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R}) \to X$, respectively.

There is an isomorphism of real vector bundles

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R}) & \xrightarrow{\Psi = 1 - ij^*} & \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \\ \downarrow{\scriptstyle p_{\mathbb{R}}} & & \downarrow{\scriptstyle p_{\mathbb{C}}} \\ X & \xrightarrow[\mathrm{id}]{} & X. \end{array}$$

Note that $\Psi^{-1}(\xi) = \Re \circ \xi$.

Go back

1. $A \to X$ is a HLA

2. For all $X^{0,1} \in \Gamma(T^{0,1}X)$ and $A^{1,0} \in \Gamma(A^{1,0})$, set

$$\nabla_{X^{0,1}} A^{1,0} = 0 \qquad \text{and} \qquad \nabla_{A^{1,0}} X^{0,1} = \mathrm{pr}^{0,1}[\rho_{\mathbb{C}} A^{1,0}, X^{0,1}].$$

   Then $(T^{0,1}X, A^{1,0})$ is a matched pair of CLAs.

3. $A_{\mathbb{C}} = A^{0,1} \bowtie A^{1,0} \longrightarrow T_X^{0,1} \bowtie A^{1,0} \longrightarrow T_X^{0,1} \bowtie T_X^{1,0} = T_{\mathbb{C}} X$

Go back

STEP 1 Given a matched pair (A, B), $A \bowtie B$ is a Lie algebroid.

Its Lie algebroid differential

$$\Gamma(\wedge^\bullet(A \oplus B)^*) \xrightarrow{d_{A\bowtie B}} \Gamma(\wedge^{\bullet+1}(A \oplus B)^*),$$

is defined by

$$(d_{A\bowtie B}\alpha)(C_0, \ldots, C_n) = \sum_{i=0}^{n}(-1)^i c(C_i)\big(\alpha(C_0, \ldots, \widehat{C}_i, \ldots, C_n)\big) + \sum_{i<j}(-1)^{i+j}\alpha([C_i, C_j], C_0, \ldots, \widehat{C}_i, \ldots, \widehat{C}_j, \ldots, C_n),$$

and satisfies $d_{A\bowtie B}^2 = 0$.

STEP 2 Now, remember that

$$\wedge^n(A \oplus B)^* = \bigoplus_{k+l=n} \wedge^k A^* \otimes \wedge^l B^*.$$

It is easy to see that

$$\begin{aligned} d_{A \bowtie B}(\Gamma(\wedge^k A^* \otimes \wedge^l B^*)) \subset \Gamma(\wedge^{k+2} A^* \otimes \wedge^{l-1} B^*) \oplus \Gamma(\wedge^{k+1} A^* \otimes \wedge^l B^*) \\ \oplus \Gamma(\wedge^k A^* \otimes \wedge^{l+1} B^*) \oplus \Gamma(\wedge^{k-1} A^* \otimes \wedge^{l+2} B^*). \end{aligned}$$

Moreover, since A and B are Lie subalgebroids of $A \bowtie B$, the stronger relation

$$d_{A \bowtie B}\Gamma(\wedge^k A^* \otimes \wedge^l B^*) \subset \Gamma(\wedge^{k+1} A^* \otimes \wedge^l B^*) \oplus \Gamma(\wedge^k A^* \otimes \wedge^{l+1} B^*)$$

holds.

Step 3 Composing $d_{A\bowtie B}$ with the natural projections on each of the direct summands

$$\begin{array}{ccc} & & \Gamma(\wedge^{k+1}A^*\otimes\wedge^l B^*) \\ & \overset{\partial_A}{\nearrow} & \uparrow \\ \Gamma(\wedge^k A^*\otimes \wedge^l B^*) & \xrightarrow{d_{A\bowtie B}} & \Gamma((\wedge^{k+1}A^*\otimes\wedge^l B^*)\oplus(\wedge^k A^*\otimes\wedge^{l+1}B^*)) \\ & \underset{(-1)^k\partial_B}{\searrow} & \downarrow \\ & & \Gamma(\wedge^k A^*\otimes\wedge^{l+1}B^*), \end{array}$$

we get two operators ∂_A and ∂_B.

From $d^2_{A\bowtie B}=0$, it follows that $\partial_A^2=0$, $\partial_B^2=0$ and $\partial_A\circ\partial_B=\partial_B\circ\partial_A$.

Go back

The operator d_π is defined by the relation

$$(d_\pi \alpha)(Y_1, \dots, Y_k) = [\pi, \alpha(Y_1, \cdots, Y_k)] + (-1)^k [\pi, Y_1 \wedge \cdots \wedge Y_k] \lrcorner \alpha,$$

where $Y_1, \dots, Y_k$ are arbitrary elements of $\mathfrak{X}^{0,1}(X)$.

Alternatively, if $\omega \in \Omega^{0,k}(X)$ and $P \in \mathfrak{X}^{l,0}(X)$, then

$$d_\pi(\omega \otimes P) = \omega \otimes [\pi, P] + \sum_{i=1}^{n} (i_{\pi^\sharp e^i} d\omega) \otimes (e_i \wedge P),$$

where $(e_1, \dots, e_n)$ is a basis of $T_x^{1,0}X$ and $(e^1, \dots, e^n)$ is the dual basis of $(T_x^{1,0}X)^*$.

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