Lax operator algebras and integrable systems

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Lax equations on Riemann surfaces

Lax equation with a spectral parameter:

$$L_t = [L, M], \quad L, M \in \mathfrak{gl}(n) \otimes \mathbb{C}[\lambda^{-1}, \lambda), \quad \lambda \in \mathcal{D}^1.$$

If one wants $\lambda \in \Sigma$ (Σ — complex Riemann surface of genus g)
then one gets a problem:
let $\deg L = -m$, $\deg M = -n$. Then $\deg[L, M] = -(m + n)$.
By Riemann-Roch theorem
the number of unknowns $= n^2(m + n - 2g + 1)$,
$(= \dim\{L\} + \dim\{M\})$, but
the number of equations $= n^2(m + n - g + 1)$
$(= \dim\{[L, M]\})$.

The system is over-determined.

Tyurin parameters of holomorphic vector bundles

Σ — complex Riemann surface.

<u>Theorem</u> (A.N.Tyurin). Every semi-stable holomorphic vector bundle $B \to \Sigma$ is characterized by the following parameters:
1) points $\gamma_1, \ldots, \gamma_{ng}$ of Σ where $n = \operatorname{rank} B$, $g = \operatorname{genus} \Sigma$;
2) $\alpha_1, \ldots, \alpha_{ng} \in \mathbb{C}P^{n-1}$.
$\Gamma(B)$ is canonically isomorphic to the space of functions $\mathcal{F} = \{\psi : \Sigma \to \mathbb{C}^n\}$ having at most simple poles at $\gamma_1, \ldots, \gamma_{ng}$ and at everyone of those points

$$\psi(z) = \frac{\alpha}{z} + O(1).$$

Lax operators (after I.M.Krichever, hep-th/0108110)

Fix Tyurin data $\gamma_1, \dots, \gamma_{ng}$, $\alpha_1, \dots, \alpha_{ng}$ and a divisor D on Σ. Consider a meromorphic function $L : \Sigma \to \mathfrak{gl}(n)$, holomorphic except at points of D, and having at most simple poles at γ's (for this reason those are called weak singularities). At every γ, let z be a local coordinate, $\alpha, \beta \in \mathbb{C}P^{r-1}$ (α is fixed, β is arbitrary) and L is of the form

$$L = \frac{L_{-1}}{z} + L_0 + O(z)$$

where

$$\boxed{L_{-1} = \alpha\beta^t} \qquad \boxed{\operatorname{tr} L_{-1} = \beta^t\alpha = 0} \qquad \boxed{L_0\alpha = k\alpha}$$

L is called a Lax operator with a spectral parameter on the Riemann surface Σ. Easy fact: $L \in \operatorname{End} B$.

Brief outline of the Hamiltonian theory(I.M.Krichever)

Let $\mathrm{D} = \sum \mathrm{m_i P_i}$ $(\mathrm{i} = 1, \ldots, \mathrm{N}, \infty)$, $\mathcal{L}^{\mathrm{D}}$ be the space of the Lax operators s.t. $\mathrm{ord_{P_i}\, L \geq -m_i}$. Take $\mathrm{a} = (\mathrm{P_i, n, m}), \mathrm{n} > 0,\ \mathrm{m} > \mathrm{m_i}$.

$\exists\,!$ (up to a scalar factor) function $\mathrm{M_a}$ of the same form as L (without any restriction on the $\mathrm{trM_a}$) which is regular outside γ's and $\mathrm{P_i}$ and has the same singular part as $\mathrm{w_i^{-m} L^n}$ at the $\mathrm{P_i}$ ($\mathrm{w_i}$ being the local coordinate). For a given L, the equations

$$\partial_{\mathrm{a}} \mathrm{L} = [\mathrm{M_a}, \mathrm{L}]$$

define a hierarchy of commuting flows on an open set of $\mathcal{L}^{\mathrm{D}}$ with Hamiltonians

$$\mathrm{H_a} = -\frac{1}{\mathrm{n}+1}\, \mathrm{res_{P_i}\, tr}\big(\mathrm{w_i^{-m} L^{n+1}}\big)\mathrm{dw_i}$$

with respect to the universal Krichever-Phong symplectic struct. on $\mathcal{L}^{\mathrm{D}}$: $\omega = \sum \mathrm{res}_{\gamma_s} \Omega \mathrm{dz} + \sum \mathrm{res_{P_i}}\, \Omega \mathrm{dz}$ where $\Omega = \delta \mathrm{tr}\big(\Psi^{-1} \mathrm{L} \delta \Psi\big)$, and Ψ is a matrix-valued function formed by eigenvectors of L.

Lie structure on Lax operators (Krichever-Sheinman)

Theorem. The space of Lax operators is closed with respect to the point-wise commutator $[L, L'](P) = [L(P), L'(P)]$ $(P \in \Sigma)$, the right hand side commutator is taken in $\mathfrak{gl}(n)$. This space is also closed with respect to the point-wise multiplication.

This means, in particular, that the commutator (the product) of Lax operators again has at most simple poles at weak singularities, and the eigenvalue condition is preserved as well as the other relations.

Definition: $\bar{\mathfrak{g}} = \{L\}$ is called the Lax operator algebra.

Examples. 1) $\alpha = 0$ (the bundle is trivial). Then $\bar{\mathfrak{g}} = \mathfrak{gl}(n) \otimes \mathcal{A}$ where $\mathcal{A}$ is the algebra of scalar-valued meromorphic functions on Σ, which are holomorphic except at P_i's, i.e. $\bar{\mathfrak{g}}$ is the Krichever-Novikov current algebra.

2) $\alpha = 0$, $g = 0$ ($\Sigma = \mathbb{C}P^1$, the bundle is trivial), $P_1 = 0$, $P_2 = \infty$. Then $\bar{\mathfrak{g}} = \mathfrak{gl}(n) \otimes \mathbb{C}[z, z^{-1}]$ — loop algebra.

Orthogonal and symplectic generalizations

$\mathfrak{g} = \mathfrak{so}(n)$:

$$L = \frac{L_{-1}}{z} + L_0 + O(z)$$

where $L_0, L_1, \ldots$ are skew-symmetric,

$$\boxed{L_{-1} = \alpha\beta^t - \beta\alpha^t} \qquad \boxed{\alpha^t\alpha = \beta^t\alpha (= \alpha^t\beta) = 0} \qquad \boxed{L_0\alpha = k\alpha}$$

$\mathfrak{g} = \mathfrak{sp}(2n)$:

$$L = \frac{L_{-2}}{z^2} + \frac{L_{-1}}{z} + L_0 + L_1 z + O(z^2)$$

where $L_{-2}, L_{-1}, L_0, L_1, \ldots$ are symplectic matrices, and

$$\boxed{L_{-1} = \alpha\beta^t + \beta\alpha^t} \qquad \boxed{\beta^t\sigma\alpha = 0} \qquad \boxed{L_0\alpha = k\alpha}$$

Notice that $\alpha^t\sigma\alpha = 0$ due to the skew-symmetry of σ.
New relations: $\boxed{L_{-2} = \nu\alpha\alpha^t \ (\nu \in \mathbb{C})}$, $\boxed{\alpha^t\sigma L_1\alpha = 0}$.

Almost graded structure (two-point case)

Define $\quad \mathfrak{g}_m := \{L \in \overline{\mathfrak{g}} \mid (L) + D_m \geq 0\}, \ m \in \mathbb{Z}$
where

$$D_m = -mP_+ + (m+g)P_- + \varepsilon \sum_{s=1}^{ng} \gamma_s,$$

and $\varepsilon = 1$ for $\mathfrak{g} = \mathfrak{sl}$, $\mathfrak{g} = \mathfrak{so}(n)$, $\varepsilon = 2$ for $\mathfrak{g} = \mathfrak{sp}(2n)$.
We call $\mathfrak{g}_m$ the (homogeneous) subspace of degree m of the Lax operator algebra $\overline{\mathfrak{g}}$.

Theorem. $\quad 1^\circ$. $\dim \mathfrak{g}_m = \dim \mathfrak{g}$.

2°. $\overline{\mathfrak{g}} = \bigoplus\limits_{m=-\infty}^{\infty} \mathfrak{g}_m$. $\quad 3^\circ$. $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \bigoplus\limits_{m=k+l}^{k+l+g} \mathfrak{g}_m$.

Corollary: $\dim \mathcal{L}^D = (\dim \mathfrak{g})(\deg D - g + 1)$.

Proof. $\quad$ Let $D = -mP_+ + (n+g)P_-$. Then $\mathcal{L}^D = \mathfrak{g}_m \oplus \ldots \oplus \mathfrak{g}_n$,

$$\dim \mathcal{L}^D = (\dim \mathfrak{g})(n - m + 1) = (\dim \mathfrak{g})(\deg D - g + 1). \qquad \square$$

(Remind: $(L) + D \geq 0$ outside γ's, hence by Riemann-Roch elements of $\mathcal{L}^D$ are defined by their singular parts on D).

Consistency of Lax equations on $\mathcal{L}^D$, $\quad D = \sum_i m_i P_i$

(the problem we addressed in the very beginning)

Lemma. $[L, M] \in T_L\mathcal{L}^D \quad \Leftrightarrow \quad ([L, M]) + D \geq 0$ outside γ's and the following equations are fulfilled at every γ:

$$z_t = -\mu^t \sigma \alpha, \quad \alpha_t = -M_0 \alpha + k\alpha \quad (*)$$

where z is a coordinate of γ (w.r.t. the initial L), σ is the matrix of the corresponding quadratic form.

($\mathfrak{gl}(n)$ — Krichever, 2001; $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$ — Sheinman, 2007).

Hence in this case there is a balance between dimensions of l.h.s. and r.h.s. of $L_t = [M, L]$, and the Lax equation is consistent.

Observe by calculation: $(*)$ is a part of the equation $L_t = [L, M]$.

Remind: by almost-gradeness theorem, the elements of $\mathcal{L}^D$ are defined by their singular parts on D.

Conclusion: Lax equation consists of the equations on Tyurin data and on singular parts of L on D (similar to $g = 0$).

Zero curvature equations with elliptic spectral parameter and the KP equation: $\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}(u_t + \frac{1}{4}(6uu_x - u_{xxx}))$

Let χ_1, χ_2 be elliptic functions s.t. $\boxed{\chi_{1t} - \chi_{2x} = [\chi_2, \chi_1]}$, $(*)$
and $(**)$ $\chi_i(\lambda) = A_i(\lambda) + O(\lambda)$ $(i = 1, 2)$ where

$$A_1(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda^{-1} - u & 0 \end{pmatrix},\ A_2(\lambda) = \begin{pmatrix} \omega_1 & \lambda^{-1} + \frac{u}{2} \\ \lambda^{-2} - \frac{u\lambda^{-1}}{2} + \omega_2 & -\omega_1 \end{pmatrix}.$$

$\omega_{1,2}$ are explicitly determined by Tyurin data:
$\omega_1 = -\frac{u_x}{4} + \frac{\wp(\gamma_1) - \wp(\gamma_2)}{\alpha_1 - \alpha_2}$, $\omega_2 = \omega_{1x} - \frac{u^2}{2} + \wp(\gamma_1) + \wp(\gamma_2)$.
Introduce $c = c(x, t)$ by $\gamma_1 = y + c(x, t)$, $\gamma_2 = y - c(x, t)$

<u>Proposition</u> (Krichever-Novikov, 1980). Every solution to $(*)$ obeying $(**)$ yields a solution to KP by

$$u(x, y, t) = \frac{c_{xx} - 1}{c_x^2} + 2\Phi c_{xx} + c_x^2(\Phi_c - \Phi^2) - \frac{c_{xxx}}{2c_x}$$

where $\Phi(y, c) = \zeta(y + c) - \zeta(y - c) - \zeta(2c)$, ζ is the Weierstrass ζ-function.

Local central extensions (Schlichenmaier-Sheinman)

Central extension: $\widehat{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathbb{C}\mathrm{t}$, $[\widehat{\mathrm{L}}, \widehat{\mathrm{L}'}] = \widehat{[\mathrm{L}, \mathrm{L}']} + \gamma(\mathrm{L}, \mathrm{L}')\mathrm{t}$, $[\mathrm{L}, \mathrm{t}] = 0$, $\forall \mathrm{L}, \mathrm{L}' \in \overline{\mathfrak{g}}$. A central extension of an almost graded Lie algebra is called local if it is also almost graded.

Theorem. For the above considered simple classic Lie algebras $\mathfrak{g}$ the corresponding Lax operator algebra has only one local central extension, up to equivalence.

The method of the proof is completely new. Known proofs are based on C.Kassel result (1984) about splitting of cocycles for Lie algebras of the form $\mathfrak{g} \otimes \mathcal{A}$, $\mathcal{A}$ being commutative, which is not the case.

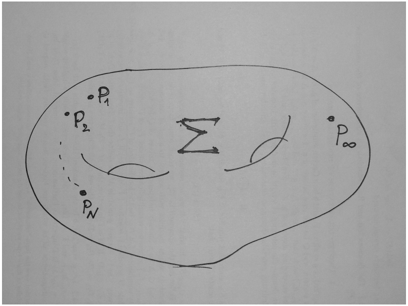
Cocycle:

$$\gamma(\mathrm{L}, \mathrm{L}') = -\operatorname{res}_{\mathrm{P}_\infty} \operatorname{tr}(\mathrm{L}d\mathrm{L}' - [\mathrm{L}, \mathrm{L}']\theta)$$

where θ is a connection form satisfying the analitic conditions similar to those for L.

Remarks on Krichever-Novikov algebras

KN algebra is a Lax operator algebra with $\alpha = 0$ ($\{\gamma\} = \{\emptyset\}$): $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A} \oplus \mathbb{C}\mathrm{t}$ where $\mathcal{A}$ is the algebra of functions on Σ, meromorphic and holomorphic outside $\mathrm{P_i}$'s. Bracket: $[\mathrm{xA}, \mathrm{yB}] = [\mathrm{x}, \mathrm{y}]\mathrm{AB} - \gamma(\mathrm{xA}, \mathrm{yB})\mathrm{t}$, $[\mathrm{xA}, \mathrm{t}] = 0$ ($\mathrm{x}, \mathrm{y} \in \mathfrak{g}$, $\mathrm{A}, \mathrm{B} \in \mathcal{A}$).



Applications:

- Deformation theory of Kac-Moody and Virasoro algebras (Fialowski-Schlichenmaier): in the ∞-dim case the formal rigidity does not exclude deformations.
- 2D CFT (Schlichenmaier-Sheinman): a most general expression for the Knizhnik-Zamolodchikov connection on $\mathcal{M}^{(1)}_{\mathrm{g,N}}$ ($\rho =$ Kodaira-Spenser mapping, T =Sugawara repr.):

$$\nabla_{\mathrm{X}} = \partial_{\mathrm{X}} + \mathrm{T}(\rho(\mathrm{X})), \quad \mathrm{X} \in \mathcal{T}\mathcal{M}^{(1)}_{\mathrm{g,N}}$$