Q-manifolds and Mackenzie Theory

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Contents

Q-manifolds

Q-manifolds are supermanifolds endowed with a homological vector field (= self-commuting odd vector field).

Features:

- A non-linear extension of the notion of a Lie algebra (together with Poisson and Schouten manifolds)
- Effective geometric language for describing algebraic structures (e.g., strongly homotopy Lie algebras, Lie algebroids, ...)

Mackenzie theory

"Mackenzie theory" is for Kirill Mackenzie. It embraces the following subjects:

- Double structures: double Lie groupoids and double Lie algebroids
- Lie bialgebroids and their "Drinfeld doubles"
- Duality theory for double and multiple vector bundles

Plan

I shall give an introduction to Q-manifold theory; in particular, examples of description of algebraic structures. I shall recall the notion of Lie algebroids. After that I shall speak about double Lie algebroids (originally introduced by Mackenzie in a very different way). I shall discuss application to a "Drinfeld double" of a Lie bialgebroid and generalizations such as multiple Lie algebroids (and multiple bialgebroids).

Graded manifolds and Q-manifolds

A graded manifold is a supermanifold with a privileged class of atlases where the coordinates are assigned weights in $\mathbb{Z}$, and the coordinate transformations are polynomial in coordinates with nonzero weights respecting the total weight. It is also assumed that the coordinates with nonzero weights run over the whole $\mathbb{R}$ (no restriction on range).
No relation between weight and parity (in general).
Example: any supermanifold (all weights are zero).
Example: the total space of a vector bundle where the coordinates on the base have zero weight, the linear coordinates on fibers are assigned weight 1.
Any graded manifold having only non-negative weights decomposes into a tower of affine fibrations, the first level being a vector bundle.

Q-manifolds

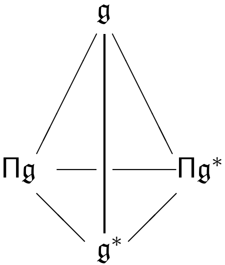
A Q-manifold is a pair (M, Q) where M is a graded manifold and $Q \in \mathfrak{X}(M)$ is an <u>odd</u> vector field such that $[Q, Q] = 0$ (equiv., $Q^2 = 0$). Q is called a homological vector field.
A morphism $(M_1, Q_2) \to (M_2, Q_2)$ is a smooth map $F \colon M_1 \to M_2$ such that $Q_1 \circ F^* = F^* \circ Q_2$.
Example: for an arbitrary manifold M define $\hat{M}$ so that $\Omega(M) = C^\infty(\hat{M})$. Then $(\hat{M}, d)$ is a Q-manifold. In coordinates $d = dx^a \frac{\partial}{\partial x^a}$.
Example: for a Lie algebra $\mathfrak{g}$ consider $\Pi\mathfrak{g}$ where Π is the parity reversion functor. Then $(\Pi\mathfrak{g}, Q)$ where $Q = \frac{1}{2}\xi^i\xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}$, is a Q-manifold. $Q^2 = 0$ is equivalent to the Jacobi identity for c_{ij}^k.

More applications of Q-manifolds

- L_∞-algebras and L_∞-morphisms
- (Non-abelian version) A_∞-algebras
- Lie algebroids and their morphisms
- Homology of Lie algebroids
- Lie bialgebroids
- (...............)

Three manifestations of a Lie algebra

Suppose $\mathfrak{g}$ is a Lie algebra. Three other equivalent manifestations:



- Linear Poisson bracket $\{x_i, x_j\} = c_{ij}^k x_k$ on $\mathfrak{g}^*$ (Berezin–Kirillov bracket)
- Linear Schouten bracket $\{\xi_i, \xi_j\} = c_{ij}^k \xi_k$ on $\Pi\mathfrak{g}^*$
- Quadratic homological vector field $Q = \frac{1}{2}\,\xi^i \xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}$ on $\Pi\mathfrak{g}$

L_∞-algebras

Consider an odd vector field $Q \in \mathfrak{X}(\mathbb{R}^{m|n})$. Let its Taylor expansion at the origin have the form

$$Q = \left(Q_0^k + \xi^i Q_i^k + \frac{1}{2}\xi^j \xi^i Q_{ij}^k + \frac{1}{3!}\xi^l \xi^j \xi^i Q_{ijl}^k + \dots \right) \frac{\partial}{\partial \xi^k}.$$

The coefficients Q_0^k, Q_i^k, Q_{ij}^k, Q_{ijl}^k, ... define a sequence of N-ary operations ($N = 0, 1, 2, 3, \dots$) on the vector space $\mathbb{R}^{n|m} = \Pi\mathbb{R}^{m|n}$, and the condition $Q^2 = 0$ expands to a linked sequence of "generalized Jacobi identities". If only the quadratic term is present, we return to the case of a Lie (super)algebra. The general case is a strong homotopy Lie algebra (L_∞-algebra)

Coordinate-free description

Given a superspace V. (For Lie algebras, $V = \mathfrak{g}$.) Each $v \in V$ defines a (constant) vector field $i_v \in V$. Define "higher derived brackets" as follows (here $N = 0, 1, 2, \ldots,$):

$$i_{\{v_1,\ldots,v_N\}_Q} := [[[\ldots[Q, v_1], v_2], \ldots, v_N](0)\,.$$

These operations odd and symmetric (in the super sense).

Theorem

They satisfy the identities

$$\sum_{k+l=N} \sum_{(k,l)\text{-shuffles}} (-1)^{\alpha}\{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}, v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\} = 0$$

for all $N = 0, 1, 2, \ldots$ if and only if $Q^2 = 0$. (Here $(-1)^{\alpha}$ is the sign prescribed by the sign rule for a permutation of homogeneous elements $v_1, \ldots, v_N \in V$.)

L_∞-morphisms

A morphism of L_∞-algebras $V_1 \to V_2$ = a morphism of the corresponding Q-manifolds (i.e., a smooth map that relates Q_1 on V_1 and Q_2 on V_2).
In coordinates: if $Q_1 = Q^k(\xi)\frac{\partial}{\partial \xi^k}$ and $Q_2 = Q^\mu(\eta)\frac{\partial}{\partial \eta^\mu}$, one has to expand

$$Q_1^i(\xi)\frac{\partial \eta^\mu}{\partial \xi^i} = Q_2^\mu(\eta(\xi))$$

into a Taylor series at the origin. (Here $F\colon (\xi^i) \mapsto (\eta^\mu(\xi))$.)

Definition of a Lie algebroid

A Lie algebroid over M is a vector bundle $E \to M$ with a Lie algebra structure on the space of sections $C^\infty(M, E)$ and a bundle map $a\colon E \to TM$ (called the anchor) satisfying

$$[u, fv] = a(u)f\,v + (-1)^{\tilde{u}\tilde{f}} f[u, v]$$

($u \in C^\infty(M, E)$ and $f \in C^\infty(M)$).
Examples: a Lie (super)algebra $\mathfrak{g}$ (here $M = \{*\}$); the tangent bundle $TM \to M$; an integrable distribution $D \subset TM$; an "action algebroid" $M \times \mathfrak{g}$.
Equivalent manifestations on "neighbors":

- Homological vector field of weight 1 on ΠE
- Poisson bracket of weight -1 on E^*
- Schouten bracket of weight -1 on ΠE^*

(structures on total spaces!).

Description via Q-manifolds

In local coordinates x^a (on the base) and ξ^i (on the fibers), we have on ΠE:

$$Q = \xi^i Q_i^a(x)\,\frac{\partial}{\partial x^a} + \frac{1}{2}\,\xi^i\xi^j Q_{ji}^k(x)\,\frac{\partial}{\partial \xi^k}\,.$$

The anchor and the Lie bracket for E are expressed by

$$a(u)f := \big[[Q, i_u)], f\big]$$

and

$$i_{[u,v]}) := (-1)^{\tilde{u}}\big[[Q, i_u], i_v\big].$$

Here the map $i\colon\ C^\infty(M,E) \to \mathfrak{X}(\Pi E)$ is $i_u = (-1)^{\tilde{u}} u^i(x)\frac{\partial}{\partial \xi^i}$.

Morphisms of Lie algebroids

The definition of a morphism of Lie algebroids over different bases (due to Higgins and Mackenzie) is tricky. It is a morphism of vector bundles

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

satisfying non-obvious conditions.

Proposition (Vaintrob)

This vector bundle map is a morphism of Lie algebroids if and only if the induced map $\Phi^{\Pi}\colon\ \Pi E_1 \to \Pi E_2$ of the opposite vector bundles is a morphism of Q-manifolds.

Homology of Lie algebroids

For a Q-manifold M, the standard cochain complex is $(C^\infty(M), Q)$.
The standard chain complex is defined as $(\mathrm{Vol}(M), L_Q)$. Here $\mathrm{Vol}(M)$ stands for the Berezin volume forms and L_Q, for the Lie derivative w.r.t. the vector field Q. Justification: correct functorial behavior w.r.t. morphisms $F\colon M_1 \to M_2$ (the existence of forward map F_*).
Pairing of chains and cochains: $\langle f, \sigma\rangle = \int_M f\sigma$ exists always.
A "Poincaré isomorphism" $(C^\infty(M), Q) \to (\mathrm{Vol}(M), L_Q)$ exists $\Leftrightarrow$ there is an invariant non-vanishing volume form ρ $\Leftrightarrow$ the cohomology "modular class" $[\mathrm{div}_{\boldsymbol{\rho}} Q] \in H(C^\infty(M), Q)$ (independent of ρ) vanishes.
For Lie algebroids one obtains $(\mathrm{Vol}(\Pi E), L_Q)$ as the chain complex. (Complex appeared in Evens, Lu, and Weinstein, 1999. Functorial property: V. Rubtsov and Th. V., in Vienna this summer.)

Definition of a Lie bialgebroid

We use the following language: a P-manifold is a Poisson manifold; an S-manifold is a Schouten manifold; a QP-manifold (a QS-manifold) possesses both Q- and P-structure (S-structure, resp.) so that the vector field is a derivation of the bracket.

Lie bialgebroids were introduced by Mackenzie and Xu; more efficient description later found by Y. Kosmann-Schwarzbach. Below is a version that uses the language of Q-manifolds.

A Lie bialgebroid over M is a Lie algebroid E over M such that E^* is also a Lie algebroid over M and so that ΠE (with the induced structure) is a QS-manifold. Equivalently: ΠE^* is a QS-manifold. (Note that there is only one type of manifestation – differently from Lie algebroids.)

Example: for $M = \{*\}$ we recover Drinfeld's Lie bialgebras.
Relevance: quantum groupoids $\Rightarrow$ Poisson groupoids $\Rightarrow$ Lie bialgebroids.

Double Lie algebroids

Double Lie algebroids were discovered by Mackenzie, who studied double Lie groupoids (in Ehresmann's sense, as groupoid objects in the category of groupoids).

$$\text{Double Lie groupoids} \Rightarrow \text{Double Lie algebroids}$$

Difficulty: no categorical definition possible; original definition is very hard. The easy part is as follows: a double Lie algebroid over M is a double vector bundle [see precise definition below]

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

such that each side (which is a vector bundle) is a Lie algebroid. The main problem is to formulate compatibility conditions.

Multiple vector bundles

A double vector bundle over M is a fiber bundle $D \to M$ with a special structure. Trivial model: $U \times V_1 \times V_2 \times V_{12}$ where V_i, V_{ij} are vector spaces and $U \subset M$. Admissible transformations: $V_1 \times V_2 \times V_{12} \to V_1 \times V_2 \times V_{12}$ that for each V_i are linear, and for V_{12} linear in V_{12} plus an extra term bilinear in $V_1 \times V_2$. In coordinates:

$$\begin{aligned} u^i &= u^{i'} T_{i'}{}^{i}, \\ w^{\alpha} &= w^{\alpha'} T_{\alpha'}{}^{\alpha}, \\ z^{\mu} &= z^{\mu'} T_{\mu'}{}^{\mu} + w^{\alpha'} u^{i'} T_{i'\alpha'}{}^{\mu} . \end{aligned}$$

In particular there is a diagram as above with sides — vector bundles. Here V_1 is the standard fiber for $A \to M$; V_2, for $B \to M$; $V_1 \times V_{12}$, for $D \to B$; and $V_2 \times V_{12}$, for $D \to A$. There is also a vector bundle $K \to M$ with the standard fiber V_{12}, called the core of the double vector bundle $D \to M$.

Everything generalizes to n-fold vector bundles.

Examples

Let $E \to M$ be an ordinary vector bundle. Then there are two associated double vector bundles (very important in differential geometry and applications):

The tangent double vector bundle

$$\begin{tikzcd} TE \arrow[r] \arrow[d, "Tp"'] & E \arrow[d, "p"] \\ TM \arrow[r] & M \end{tikzcd}$$

The core is isomorphic to $E \to M$.

The cotangent double vector bundle

$$\begin{tikzcd} T^*E \arrow[r] \arrow[d] & E \arrow[d] \\ E^* \arrow[r] & M \end{tikzcd}$$

The core bundle in this case is $T^*M \to M$.

Duality for multiple bundles

Duality theory is due to Mackenzie (and independently to Konieczna–Urbanski). Main statements:

- $D_A^* \to A$ extends to a double vector bundle over M, with the new side $K^* \to M$ and the new core $B^* \to M$
- $D_A^* \to K^*$ and $D_B^* \to K^*$ are canonically dual as vector bundles over K^* (best understood in coordinates; the duality is given by the invariant form

$$u^i u_i - w^\alpha w_\alpha$$

where the minus is absolutely essential!)

There is a 'cornering' (instead of 'pairing'):

$$\begin{tikzcd}
 & & & D_B^* \arrow[dl] \arrow[dd] \\
D_A^* \arrow[rr] \arrow[dd] & & K^* \arrow[dd] & \\
 & D \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
A \arrow[rr] & & M &
\end{tikzcd}$$

Neighbors of a pre- double Lie algebroid

$$
\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_A D \arrow[r] \arrow[d] & \Pi B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_A D_A^* \arrow[r] \arrow[d] & \Pi K^* \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_{K^*} D_B^* \arrow[r] \arrow[d] & \Pi B \arrow[d] \\
K^* \arrow[r] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
D_A^* \arrow[r] \arrow[d] & K^* \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_B D \arrow[r] \arrow[d] & B \arrow[d] \\
\Pi A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_B D_B^* \arrow[r] \arrow[d] & B \arrow[d] \\
\Pi K^* \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi^2 D_A^* \arrow[r] \arrow[d] & \Pi K^* \arrow[d] \\
\Pi A \arrow[r] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
D_B^* \arrow[r] \arrow[d] & B \arrow[d] \\
K^* \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi^2 D \arrow[r] \arrow[d] & \Pi B \arrow[d] \\
\Pi A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi_{K^*} D_A^* \arrow[r] \arrow[d] & K^* \arrow[d] \\
\Pi A \arrow[r] & M
\end{tikzcd}
\quad
\begin{tikzcd}
\Pi^2 D_B^* \arrow[r] \arrow[d] & \Pi B \arrow[d] \\
\Pi K^* \arrow[r] & M
\end{tikzcd}
$$

Neighbors with structure on total space

Two homological fields of weights $(1,0)$ and $(0,1)$:

$$\begin{array}{ccc} \Pi^2 D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}$$

A Poisson or Schouten bracket of weight $(-1,-1)$ and a homological vector field of weight $(1,0)$ or $0,1$:

$$\begin{array}{ccccc} \Pi_{K^*} D_A^* & \longrightarrow & K^* \Pi_{K^*} D_B^* & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \quad \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \quad K^* & \longrightarrow & M \end{array}$$

$$\begin{array}{ccccc} \Pi^2 D_A^* & \longrightarrow & \Pi K^* \Pi^2 D_B^* & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \quad \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \quad \Pi K^* & \longrightarrow & M \end{array}$$

Main theorem

Compatibility condition for the first diagram: commutativity. Compatibility condition for the last four diagrams: derivation property w.r.t. the bracket.

Theorem

All five conditions are equivalent. The last four conditions are the different ways of saying that (D_A^*, D_B^*) is a Lie bialgebroid over K^*.

Remark: that (D_A^*, D_B^*) is a Lie bialgebroid over K^* is the crucial part of Mackenzie's definition of a double Lie algebroid.

Corollary

The double vector bundle $D \to M$ is a double Lie algebroid if and only if the homological vector fields on $\Pi^2 D \to M$ commute.

Extension to the higher case: an n-fold Lie antialgebroid is an n-fold vector bundle $E \to M$ with n commuting homological vector fields Q_i of weights δ_{ij}. Then $\Pi^n E \to M$ is an n-fold Lie algebroid, and vice versa.

Drinfeld double of a Lie bialgebroid

According to Mackenzie: a double Lie algebroid

$$\begin{array}{ccc} T^*E = T^*E^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

According to Roytenberg: a Q-manifold with homological field $Q = X_{H_E} + X_{H_{E^*}}$:

$$\begin{array}{ccc} T^*\Pi E = T^*\Pi E^* & \longrightarrow & \Pi E^* \\ \downarrow & & \downarrow \\ \Pi E & \longrightarrow & M \end{array} \tag{1}$$

<u>Statement:</u> these pictures are identical up to change of parity.

More on doubles

General principle

Taking the double of an n-fold Lie bialgebroid should give an $(n+1)$-fold Lie bialgebroid, with an additional property, such as a symplectic structure.

References

arXiv:math.DG/0608111

arXiv:0709.4232 [math.DG]