Something in the overlap of
Algebra, Geometry, and Physics

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1. Lie Algebroid YM-theories
2. Higher form gauge theories

} <u>Yang-Mills</u> <u>type</u> <u>Sigma Models</u>

3. Current Algebras and (Super) Geometry

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Physics $\rightsquigarrow$ Sigma Models

(Σ, h) ... d-dim (pseudo-)Riemannian spacetime

n 0-forms ("scalar fields")

$$\underbrace{S[\phi^a] = \int_\Sigma d\phi^a \wedge * d\phi^a}, \quad (a = 1, \dots, n)$$

$\phi^a \in C^\infty(\Sigma) \longleftrightarrow \varphi : \Sigma \to \mathbb{R}^n$

Sigma model: replace by mfd. M with geom.

e.g. (M, g): $\quad X : \Sigma \to M, \quad X^i = X^*(x^i) \quad (i = 1, \dots, n)$

$$\boxed{S[X] = \frac{1}{2}\int_\Sigma \|dX\|^2}, \quad \|dX\|^2 \equiv (X^* g_{ij})\, dX^i \wedge * dX^j$$

n 1-forms ("gauge fields") $\underbrace{S[A^a] = \int_\Sigma dA^a \wedge * dA^a}$

deform. theory $\rightsquigarrow$ Yang-Mills $\quad A \in \Omega^1(\Sigma, \mathfrak{g})$

$F = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(\Sigma, \mathfrak{g})$, $(\mathfrak{g}, {}^{\mathfrak{g}}g)$ quadr. Lie algebra

$$\underbrace{S_{YM}[A] = \frac{1}{2}\int_\Sigma \|F\|^2}, \quad \|F\|^2 \equiv {}^{\mathfrak{g}}g(F \wedge * F)$$

gauge invar.: $\delta_\varepsilon A = d\varepsilon + [A, \varepsilon], \quad \varepsilon \in C^\infty(\Sigma, \mathfrak{g})$

$A \in \Omega^1(\Sigma, \mathfrak{g}) \longleftrightarrow a : T\Sigma \to \mathfrak{g}$

goal: replace by Lie algebroid E with geom.

Lie algebroid geometry.

Def.: Lie algebroid — vector bundle $\begin{array}{ccc} E & \xrightarrow{\rho} & TM \\ \downarrow & \swarrow & \\ M & & \end{array}$

$[\,,\,]$ Lie algebra on $\Gamma(E)$ s.t.

$$[\psi_1, f\psi_2] = f[\psi_1,\psi_2] + \rho(\psi_1)f\cdot\psi_2$$

Examples: • $M = pt. \Rightarrow E \cong \mathfrak{g}$, Lie algebra • $\underline{E = TM}$, "standard" • $E = T^*M$, M Poisson

• "exterior calculus": ${}^E\Omega^\bullet = \Gamma(\Lambda^\bullet E^*)$, ${}^E d : {}^E\Omega^k \to {}^E\Omega^{k+1}$,

$${}^E d \circ {}^E d = 0$$

• E-Lie derivative: ${}^E\mathcal{L}_{\psi_1}(\psi_2) = [\psi_1,\psi_2]$ etc.

• E-covariant derivative: $E \searrow M \swarrow V$

$${}^E\nabla_\psi(v) \in \Gamma(V), \quad {}^E\nabla_\psi(fv) = f\,{}^E\nabla_\psi(v) + \rho(\psi)f\cdot v,$$

E-curvature ${}^E R$, E-torsion ${}^E T$, etc.

e.g. ∇ conn. on E $\longrightarrow$ ${}^E\nabla_{\psi_1}(\psi_2) = \nabla_{\rho(\psi_1)}\psi_2$

$\longrightarrow$ ${}^E\tilde{\nabla}_{\psi_1}(\psi_2) = \nabla_{\rho(\psi_2)}\psi_1 + [\psi_1,\psi_2]$

• adinv. metrics on $\mathfrak{g}$ $\quad$ (${}^E g$ ... fiber metric on E)

$\swarrow$ OR $\searrow$

$\underline{{}^E\tilde{\nabla}\,{}^E g = 0}$ (1) $\qquad$ $\underline{{}^E\mathcal{L}_{\psi_k}\,{}^E g = 0}$ (2) $\quad \langle \psi_k|_x\rangle_k = E_x$

The new setting:

- $$\begin{array}{ccc} T\Sigma & \xrightarrow{a} & E \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & M \end{array} \quad \longleftrightarrow \quad \left.\begin{array}{l} A \in \Omega^1(\Sigma, X^*E) \\ X : \Sigma \to M \end{array}\right\}$$ "gauge fields"

- $$\left.\begin{array}{l} F_{(1)} = dX - \rho(A) \in \Omega^1(\Sigma, X^*TM) \\ F_{(2)} = DA - \frac{1}{2}\,{}^E T(A \wedge A) \in \Omega^2(\Sigma, X^*E) \end{array}\right\}$$ "field strengths"

using some ∇ on E, E-torsion of ${}^E\nabla_\cdot = \nabla_{\rho(\cdot)}$

- (inf.) gauge transformations: (tricky, cf. also M. Bojowald, A. Kotov, T.S. '04)

$$\delta_\varepsilon X = \rho(\varepsilon) \in \Gamma(X^*TM) \qquad (\delta_\varepsilon X^i = \rho^i_a \varepsilon^a)$$

$$\varepsilon \in \Gamma(\Sigma, X^*E)$$

$x^i \in C^\infty(M|_U)$, $\xi_a \in \Gamma(E|_U)$: $\rho(\xi_a) = \rho^i_a \frac{\partial}{\partial x^i}$, $[\xi_a, \xi_b] = C^c_{ab}\,\xi_c$

$$\delta^{(0)}_\varepsilon A^a = d\varepsilon^a + C^a_{bc} A^b \varepsilon^c \qquad \text{frame dependent } ⚡$$

→ version 1: $\delta^{(1)}_\varepsilon A^a = \delta^{(0)}_\varepsilon A^a + \Gamma^a_{ib}\varepsilon^b F^i$, $F_{(1)} = F^i \partial_i$

→ version 2: $\epsilon \in C^\infty(\Sigma) \otimes \Gamma(E)$, $\epsilon = \epsilon^a(\sigma, x)\,\xi_a$

$$\varepsilon^a = X^*\epsilon^a, \quad \delta^{(2)}_\varepsilon A^a = \delta^{(0)}_\varepsilon A^a - X^*(\epsilon^a_{,i})\, F^i$$

$$\left(\epsilon^a \in C^\infty(\Sigma \times M) \cong C^\infty(\Sigma) \otimes C^\infty(M)\right)$$

adding "matter" or "scalar fields":

- $\begin{tikzcd} E \arrow[dr] & & V \arrow[dl, "\pi"'] \\ & M & \end{tikzcd}$, ${}^E\nabla$ flat E-connection on V

$\phi \in C^\infty(\Sigma, X^*V)$, $\phi = \phi^I v_I$

$$S = S_{LAYM}[a,\lambda] + S_{matter}[\phi, a]$$

$$S_{matter}[\phi, a] = \frac{1}{2}\int_\Sigma (X^*{}^Vg)(D_A\phi \wedge * D_A\phi) ,$$

${}^E\nabla\, {}^Vg = 0$ $\quad \left(D_A\Phi^I = d\Phi^I + \Gamma^I_{aJ} A^a \Phi^J \right)$

$\hookrightarrow$ gauge invariance persists

- make sigma model for ϕ-fields:

again a Lie algd.

$$\begin{tikzcd} \pi^*E \arrow[r] \arrow[d] & E \arrow[d] \\ V \arrow[r, "\pi"] & M \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{E} \arrow[r] \arrow[d] & E \arrow[d] \\ \tilde{M} \arrow[r, "\pi"] & M \end{tikzcd}$$

a part. Q-bdle.

$$\boxed{S_{LAYM + matter}[\tilde{a}, \lambda] = \int_\Sigma \langle \lambda \wedge \pi_* \tilde{F}_{(1)} \rangle + \frac{1}{2} \| \tilde{F}_{(2)} \|^2 + \frac{1}{2} \tilde{g} \left(\tilde{F}^{vert}_{(1)} \wedge * \tilde{F}^{vert}_{(1)} \right)}$$

(now nonlin. in ϕ^I; $(\tilde{X}^i) = (X^i, \Phi^I)$)

Invariant action functional:

1^{st} (natural) attempt: $S_0 = \frac{1}{2}\int \|F_{(1)}\|^2 + \|F_{(2)}\|^2$

using metric g on M, E_g on E

Thm.: (No-go) S_0 invar. w.r.t. $\delta_\epsilon^{(1)} \Leftrightarrow E \cong M \times \mathfrak{g}$ Lie algeb. action

($F_{(2)} \sim F_A$, $F_{(1)} \sim D_A X$: YM+"Higgs")

Way out:

$$\boxed{S_{LAYM}[a,\lambda] = \int_\Sigma \langle \lambda, F_{(1)} \rangle + \frac{1}{2}\|F_{(2)}\|^2}$$

where $\lambda \in \Omega^{d-1}(\Sigma, X^*T^*M)$ T.S. of, $\|F_{(2)}\|^2 = (X^*E_g)(F_{(2)} \wedge * F_{(2)})$

S_{LAYM} gauge inv. w.r.t.
- $\delta_\epsilon^{(1)} \Leftarrow \nabla^{E(\nabla)} E_g = 0$ (1)
- $\delta_\epsilon^{(2)} \Leftarrow \mathcal{L}^E_{\psi_\epsilon} E_g = 0$ (2)

(with $\epsilon = \epsilon^a(\sigma)\psi_a$!)

(?) ↝ "physics": classical moduli space of solutions for $\Sigma \cong \mathbb{R}^d$

$\mathfrak{g}$ isotropy Lie algebra of orbit $\mathcal{O} \subset M$

$\mathfrak{g}'$ $\mathcal{O}' \subset M$

I. From Bianchi to Q-manifolds

gauge fields: $X^i, A^a, B^B, \ldots$ } A^α (collectively)

form deg. $0, 1, 2, \ldots, p$

$i = 1, \ldots, n$, $a = 1, \ldots, r$, $B = 1, \ldots, s$

on Σ spacetime

field strengths: (general ansatz)

$$F^i = dX^i - \rho^i_a A^a$$

$$F^a = dA^a + \tfrac{1}{2} C^a_{bc} A^b A^c - t^a_B B^B$$

$$F^B = dB^B + \Gamma^B_{aC} A^a B^C - \tfrac{1}{6} H^B_{abc} A^a A^b A^c + \ldots$$

structural fcns (dep. on X)

collectively $F^\alpha = dA^\alpha + \ldots$

generates ideal $\mathcal{I}$ in algebra $\langle A^\alpha, dA^\alpha \rangle$

the condition (Bianchi):

$$dF^\alpha \stackrel{!}{=} \lambda^\alpha_\beta F^\beta \quad \Leftrightarrow \quad \underline{d\mathcal{I} \subset \mathcal{I}}$$

field dependent

(e.g. in ordinary YM: $dF^a + C^a_{bc} A^b F^c = 0$

OR when X^i some matter, $F^i \sim DX^i$ etc) recall: A Q-manifold is a $\mathbb{Z}_{(\geq 0)}$-graded manifold $\mathcal{M}$ with a homological, degree one vector field Q.

example: $\mathcal{M}_1 = T[1]\Sigma$ with $Q_1 = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

coordinates σ^μ, $\theta^\mu \equiv d\sigma^\mu$ $\quad Q_1^2 \equiv 0$

deg. $0 \quad 1$

$\left(C^\infty(T[1]\Sigma) \cong \Omega^\bullet(\Sigma) \,,\ Q_1 \cong d \ \text{de Rham} \right)$

define: $\mathcal{M}_2 = M \times \mathbb{V}[1] \times \mathbb{W}[2] \times \ldots \times \mathbb{U}[p]$

coordinates $x^i \quad \xi^a \quad b^B \quad \ldots \quad \} \ q^\alpha$ (collectively)

deg $0 \quad 1 \quad 2 \quad \ldots$

set $Q_2 := \rho_a^i \xi^a \frac{\partial}{\partial x^i} - \frac{1}{2} C^a_{bc} \xi^b \xi^c \frac{\partial}{\partial \xi^a} + t^a_B b^B \frac{\partial}{\partial \xi^a} -$

$$- \Gamma^B_{aC} \xi^a b^C \frac{\partial}{\partial b^B} + \frac{1}{6} H^B_{abc} \xi^a \xi^b \xi^c \frac{\partial}{\partial b^B} + \ldots$$

(most general deg $+1$ vect. field, above coeff. fcns)

Thm.: $\dim \Sigma \geq p+2$,

Bianchi $dI \subset I \iff Q_2^2 = 0$, $\mathcal{M}_2$ Q-mfd

Proof: gauge fields $\overset{1:1}{\longleftrightarrow}$ map $a : \mathcal{M}_1 \to \mathcal{M}_2$ degr. preserving $(\mathcal{M}_1 \equiv T[1]\Sigma)$

s.t. $A^\alpha \equiv a^*(q^\alpha)$

i.e. $X^i = a^*(x^i)$, $A^a = a^*(\xi^a)$, $B^\beta = a^*(b^\beta)$, ...

$\left(\begin{matrix} \updownarrow \\ \text{map } \Sigma \to M \end{matrix}\right)$

then $F^\alpha \equiv \mathcal{F}(q^\alpha)$ with $\mathcal{F} = Q_1 \circ a^* - a^* \circ Q_2$

obviously $Q_1 \circ \mathcal{F} \equiv \underbrace{- \mathcal{F} \circ Q_2}_{\text{result in } \mathcal{I}} \underbrace{- a^* \circ Q_2^2}_{\text{needs to vanish}}$ #

Use for deriving explicit form of Bianchi:

(note: $\mathcal{F}(\phi) = F^\alpha a^*(\partial_\alpha \phi)$ "Leibniz")

$(Q_2^2 = 0)$ ← the "heavy calculation" one saves

e.g.: $dF^i = (Q_1 \circ \mathcal{F})(x^i) = - \mathcal{F} \circ Q_2(x^i) =$

$= - \mathcal{F}(\rho^i_a \xi^a) = - F^j \rho^i_{a,j} A^a - \rho^i_a F^a.$

etc.

encoded geometry:

- degree 1 Q-mfd $\overset{1:1}{\longleftrightarrow}$ Lie algebroid $\begin{array}{c} E \\ \downarrow \\ M \end{array}$

Thm.:

degree 2 Q-mfd (with splitting) $\overset{1:1}{\longleftrightarrow}$
$$\begin{tikzcd} F \arrow[rr, "t"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$

almost Lie algebroids

${}^{E}\nabla$ ... E-con. on F

$H \in {}^{E}\Omega^3(M,F)$ etc.

"anomaly"

(special case: semi-strict Lie-2-algebra ($M = pt$))

II. Degree 2 Q-manifolds

degree 2 supermfd. $\mathcal{M} \underset{\text{locally}}{\simeq} U \times \mathbb{V}[1] \times \mathbb{W}[2]$

x^i, $i = 1, \dots, n \equiv \dim M$

ξ^a, $a = 1, \dots, r \equiv \dim \mathbb{V}$

b^B, $B = 1, \dots, s \equiv \dim \mathbb{W}$

most general degree 1 super-vector field Q:

$$Q = \rho^i_a \xi^a \frac{\partial}{\partial x^i} - \frac{1}{2} C^a_{bc} \xi^b \xi^c \frac{\partial}{\partial \xi^a} + t^a_B b^B \frac{\partial}{\partial \xi^a} - \Gamma^C_{aB} \xi^a b^B \frac{\partial}{\partial b^C} + \frac{1}{6} H^B_{abc} \xi^a \xi^b \xi^c \frac{\partial}{\partial b^B}$$

$Q^2 \overset{!}{=} 0$:

$$Q^2 x^i = 0 \Leftrightarrow \begin{cases} \rho^j_{[a} \rho^i_{b],j} - \frac{1}{2} \rho^i_c C^c_{ab} = 0 \\ \rho^i_a t^a_B = 0 \end{cases}$$

$$Q^2 \xi^a = 0 \Leftrightarrow \begin{cases} C^e_{[ab} C^d_{c]e} + \rho^i_{[c} C^d_{ab],i} - \frac{1}{3} t^d_D H^D_{abc} = 0 \\ \rho^i_{[b} \Gamma^B_{a]D,i} + \frac{1}{2} C^e_{ab} \Gamma^B_{eD} - \Gamma^B_{[aC} \Gamma^C_{b]D} + \frac{1}{2} H^B_{abc} t^c_D = 0 \\ \rho^i_c t^a_{B,i} - C^a_{bc} t^b_B - \Gamma^D_{cB} t^a_D = 0 \end{cases}$$

$$Q^2 b^B = 0 \Leftrightarrow \begin{cases} t^a_{(A} \Gamma^B_{aC)} = 0 \\ \rho^i_{[a} H^B_{bcd],i} - \frac{3}{2} C^e_{[ab} H^B_{cd]e} + \Gamma^B_{[a|D} H^D_{|bcd]} = 0 \end{cases}$$

encoded geometry:

- $n=0$, $s=0$: $(\mathcal{M}, Q) \cong (\mathfrak{g}[1], d_{CE})$

  degree 1 Q-vector space $\overset{1:1}{\longleftrightarrow}$ Lie algebra

- $s=0$: $(\mathcal{M}, Q) \cong (E[1], {}^{E}d)$,

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "\rho"] & \\ M & TM \arrow[l] \end{tikzcd}$$

  $[\cdot,\cdot]$ Lie bracket on $\Gamma(E)$

  $[\psi_1, f\psi_2] = f[\psi_1, \psi_2] + (\rho(\psi_1)f)\psi_2$ Leibniz

  degree 1 Q-mfd $\overset{1:1}{\longleftrightarrow}$ Lie algebroid

- **Thm.:** degree 2 Q-mfd with splitting $\overset{1:1}{\longleftrightarrow}$

almost Lie algebroids

$$\begin{tikzcd} & F \arrow[r, "t"] \arrow[dl, "0"'] \arrow[d] & E \arrow[dl] \arrow[d, "\rho"] \\ TM \arrow[r] & M & TM \arrow[l] \end{tikzcd}$$

$\oplus$ E-conn. on F: ${}^{E}D$

& $H \in {}^{E}\Omega^3(M, F)$

"anomaly"

s.t.: $t({}^{E}D_\psi v) = [t(v), \psi]_E$ & ${}^{E}D_{t(v)} w = [v,w]_F$

Jacobiators: in F … $H \circ t$

in E … $t \circ H$

curvature of conn.: ${}^{E}R(\cdot,\cdot)v = H(\cdot,\cdot,t(v))$

also ${}^{E}D_\psi([v,w]_F) = \ldots + H(\psi, t(v), t(w))$. & ${}^{E}DH = 0$

<u>Cor. 1:</u> degree 2 Q-mfd w. splitting & $H=0$ $\overset{1:1}{\longleftrightarrow}$

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[dr] & E \arrow[d] \arrow[dr, "\rho"] & \\ & M & TM \arrow[l] \end{tikzcd}$$

Lie algebroid

bundle of Lie algebras

$\oplus$ E-connection on F (repr.):

$${}^E D : {}^E\Omega^\bullet(M,F) \to {}^E\Omega^{\bullet+1}(M,F)$$

s.t.: $t\left({}^E D_\psi v\right) = [t(v), \psi]_E$

& ${}^E D_{t(v)} w = [v, w]_F$

<u>Cor. 2:</u> degree 2 Q-vector sp. w. splitting & $H=0$

$\overset{1:1}{\longleftrightarrow}$ Lie-2-algebra

$$\left({}^E\Omega^p(M,F) \equiv \Gamma(\Lambda^p E^* \otimes F) \right)$$

<u>Thm.</u> (Roytenberg):

symplectic deg 2 Q-mfd $\overset{1:1}{\longleftrightarrow}$ Courant algebroid

$\Downarrow$

$s = n$

$b^B \sim p_i$

$$\omega = dx^i \wedge dp_i + \tfrac{1}{2} \kappa_{ab}\, d\xi^a \wedge d\xi^b$$

fiber metric on E

IV. Gauge transformations

start as physicist again:

apriori new structural fcns. (of X)

$$\delta_\varepsilon X^i = \bar{\rho}^i_a \varepsilon^a$$

$$\delta_\varepsilon A^a = d\varepsilon^a + \bar{C}^a_{bc} A^b \varepsilon^c - \bar{t}^a_B \varepsilon^B + O(F^i)$$

$$\delta_\varepsilon B^B = d\varepsilon^B - \bar{\Gamma}^B_{ac} \varepsilon^a B^c + \bar{\Gamma}'^B_{ac} A^a \varepsilon^c + \frac{1}{2} \bar{H}^B_{abc} A^a A^b \varepsilon^c + O(F^i, F^a)$$

(for $p=2$)

the condition

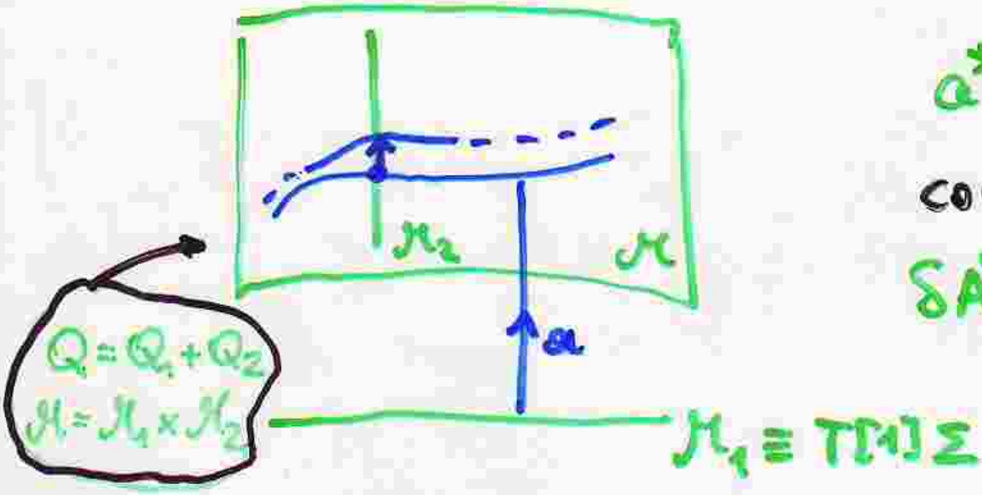
$$\delta_\varepsilon I \overset{!}{\subset} I \quad (*)$$

Thm.: $\dim \Sigma \geq 3$, I s.t. $dI \subset I$, then

$$(*) \iff \bar{\rho}^i_a = \rho^i_a, \ldots$$
$$\bar{\Gamma}^B_{ac} = \bar{\Gamma}'^B_{ac} = \Gamma^B_{ac}, \ldots \qquad \square$$

The $O(F^\alpha)$ contributions may depend on details of the theory (action fctl.)

for main part $\exists$ elegant reformulation:



$a^*(q^\alpha) = A^\alpha$

consider

$\delta A^\alpha = (a^* \circ V)(q^\alpha)$

$V \in \mathfrak{X}^0_{vert}(\mathcal{M})$

$$\Rightarrow \underline{\delta F^\alpha} = \left(Q_1 \circ \underbrace{a^* \circ V} - \underbrace{a^* \circ V} \circ Q\right)(q^\alpha)$$

$$\equiv \mathcal{F}(V(q^\alpha)) + \left(a^* \circ d_Q(V)\right)(q^\alpha)$$

where $d_Q(V) = [Q, V]$

Note: $\left(\mathfrak{X}^\bullet_{vert}(\mathcal{M}), d_Q\right)$ a complex

starts with deg $-p$ ($p = \deg \mathcal{M}_2$)

$\left(d_Q^2 = 0 \Leftrightarrow \frac{1}{2}[Q,Q] = Q^2 = 0\right)$

solution above: $V = d_Q(\varepsilon) \equiv [Q, \varepsilon]$

where $\varepsilon \in \mathfrak{X}^{-1}_{vert}(\mathcal{M})$

$$\varepsilon = \varepsilon^a \frac{\partial}{\partial \xi^a} + \varepsilon^B \frac{\partial}{\partial b^B} \underbrace{+ \ldots}_{\text{absent for } p=2}$$

Thm.: Infinitesimal gauge transformations

(~ "main part" - in some frame)

form a Lie algebra isomorphic to

$\left(\mathfrak{X}^{-1}_{vert}(\mathcal{M}) / \ker d_Q \,,\, [\cdot,\cdot]_Q \right)$ where

$[\varepsilon_1, \varepsilon_2]_Q = [d_Q(\varepsilon_1), \varepsilon_2] \equiv [[\varepsilon_1, Q], \varepsilon_2]$ derived bracket

(**Remark:** an "open algebra" in usual setting !)

V. Gauge-invariant YM-type Action

(e.g. $p=2$)

$$\boxed{S[\underbrace{X, A, B}_{a}; \lambda, \bar{\lambda}] = \int_\Sigma \langle \lambda \wedge F_{(1)} \rangle + \langle \bar{\lambda} \wedge F_{(2)} \rangle + \tfrac{1}{2} \| F_{(3)} \|^2}$$

T.S. '05 (unpubl.)

$\lambda \in \Omega^{d-1}(\Sigma, X^* T^* M)$ $\qquad$ $\bar{\lambda} \in \Omega^{d-2}(\Sigma, X^* E^*)$

where $\| F_{(3)} \|^2 \equiv {}^F g(F_{(3)} \wedge * F_{(3)})$

and $\quad \underline{{}^E\nabla \, {}^F g = 0}$

$$\left(\begin{array}{ccc} F & & E \\ & \searrow \ \swarrow & \\ & M & \end{array} \,,\ {}^E\nabla \text{ an } E\text{-connection on } F \right)$$

Current Algebras and (Super-) Geometry

• $T^*LM =: \left\{ \begin{array}{ccc} p: TS^1 & \to & T^*M \\ \downarrow & & \downarrow \\ x: S^1 & \to & M \end{array}\right\} \ni \left[\begin{array}{l} p\in\Omega^1(S^1, x^*T^*M) \\ x: S^1\to M\end{array}\right.$

$$\left.\begin{array}{l} H\in\Omega^3_{cl}(M), \quad ev: S^1\times LM\to M \\ \qquad\qquad\qquad (\sigma, x)\mapsto x(\sigma)\end{array}\right\} \to \underbrace{\omega = \omega_{canonical} + \int_{S^1} ev^*H}_{\text{symplectic form}}$$

$f\in C^\infty(M)$, $\mu\in\Omega^1(S^1)$ $\to$ $J^{(0)}_f[\mu] := \int_{S^1} x^*f\cdot\mu \in \mathcal{F}(LM)\subset\mathcal{F}(T^*LM)$

$\alpha\in\Omega^1(M)$, $\varphi\in C^\infty(S^1)$ $\to$ $J^{(1)}_\psi[\varphi] := \int_{S^1}\varphi\,(p^*v + x^*\alpha) \in \mathcal{F}(T^*LM)$

$v\in\mathfrak{X}(M)$, $\psi = v+\alpha\in\Gamma(TM\oplus T^*M)$

Note: $J^{(0)}_f[d\varphi] = -\int_{S^1} x^*df\cdot\varphi = -J^{(1)}_{df}[\varphi]$ (1)

$$\boxed{\begin{array}{l}\{J^{(0)}_{f_1}[\mu_1], J^{(0)}_{f_2}[\mu_2]\} = 0, \quad \{J^{(1)}_\psi[\varphi], J^{(0)}_f[\mu]\} = J^{(0)}_{\rho(\psi)f}[\varphi\mu] \\ \{J^{(1)}_{\psi_1}[\varphi_1], J^{(1)}_{\psi_2}[\varphi_2]\} = J^{(1)}_{[\psi_1,\psi_2]}[\varphi_1\varphi_2] + J^{(0)}_{(\psi_1,\psi_2)}[\varphi_2\, d\varphi_1]\end{array}}$$

A. Alekseev, T.S. '04 (2) "anomaly"

read off:

$\rho(\psi)\equiv pr_1(v\oplus\alpha) = v$, $(\psi_1,\psi_2)\equiv\langle\alpha_2, v_1\rangle + \langle\alpha_1, v_2\rangle$

$[\psi_1,\psi_2]\equiv[v_1,v_2]\oplus\mathcal{L}_{v_1}\alpha_2 - \iota_{v_2}\alpha_1 + \iota_{v_1}\iota_{v_2}H$

(twisted) Courant bracket

exact Courant algebroid with splitting

$$\begin{array}{ccc} E\cong TM\oplus T^*M & & \\ \downarrow & \searrow^{\rho} & \\ M & \leftarrow & TM \end{array}$$

- **Examples:**
  - $M = G$ Lie group $\quad \psi = v + \alpha$ right (left) invar.

    $\Rightarrow \{J^{(1)}_{\cdot}, J^{(1)}_{\cdot}\} = \ldots$ Kac Moody algebra
  - $J^{(1)}_{\pi^\#(\alpha)+\alpha}$, $\alpha \in \Omega^1(M)$, $\pi \in \mathfrak{X}^2(M)$ constraints in the (twisted) Poisson sigma model
  - Noether currents of ordinary sigma models w. WZ-term

- **Involutivity of $J^{(1)}$ currents:** $D \subset E$

$$\left.\begin{array}{l} J^{(1)}_\psi[\varphi],\ \psi \in \Gamma(D) \\ \text{involutive w.r.t. } \{\ ,\ \}_{T^*LM} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} [\Gamma(D), \Gamma(D)] \subset \Gamma(D) \quad \leftarrow \text{Courant br.} \\ D \text{ isotropic w.r.t. } (\cdot,\cdot) \end{array}\right.$$

when maximal: **Dirac structure**

$\cup$ (twisted) Poisson $\quad$ $\cup$ (twisted) praesympl.

(twisted) symplectic

- Sigma model s.t. J_ψ, $\psi \in \Gamma(D)$, D max. isotrop. are its constraints: Dirac sigma models

  $\cup$

  twisted PSM, G/G WZW, ...

A. Kotov, P. Schaller, T.S. '04

- all Courant algebroid axioms contained in current alg.

e.g. $\underline{0 = \{ J^{(1)}_\psi[\varphi], J^{(1)}_\psi[\varphi] \}} = J^{(1)}_{[\psi,\psi]}[\varphi^2] + J^{(0)}_{(\psi,\psi)}[\varphi d\varphi]$

$$J^{(0)}_{(\psi,\psi)}[\varphi d\varphi] = \tfrac{1}{2} J^{(0)}_{(\psi,\psi)}[d(\varphi^2)] = - J^{(1)}_{\frac{1}{2}d(\psi,\psi)}[\varphi^2]$$

$\Rightarrow \quad \underline{[\psi,\psi] = \tfrac{1}{2} d(\psi,\psi)}$ etc

Thm.: current algebra "of this form" $\overset{1:1}{\longleftrightarrow}$ (degenerate) Courant algebroids

(cf. fiber lin. Poisson str. on a v.b. E^* $\overset{1:1}{\longleftrightarrow}$ Lie algebroid str. on E)

- Replace S^1 by higher dimensional Σ, e.g. Σ_2

- $\left\{ \begin{array}{ccc} T[1]\Sigma_2 & \rightarrow & T^*[2]M \oplus E[1] \\ \downarrow & & \downarrow \\ \Sigma_2 & \longrightarrow & M \end{array} \right\} \ni \begin{array}{l} p \in \Omega^2(\Sigma_2, X^*T^*M) \\ X : \Sigma_2 \to M \\ A \in \Omega^1(\Sigma_2, X^*E) \end{array}$

- $\omega = \int_{\Sigma_2} \delta p_i \wedge \delta X^i + {}^E g_{ab}\, \delta A^a \wedge \delta A^b + ev^* H, \quad H \in \Omega^4_{cl}(M)$

- $J^{(0)}_f[\lambda] = \int_{\Sigma_2} X^* f \cdot \lambda, \; J^{(1)}_\cdot[\mu] = \ldots, \; J^{(2)}_\cdot[\varphi] = \int_{\Sigma_2} \varphi \cdot v^i p_i + \ldots$

etc. $\underset{e.g.}{\rightsquigarrow}$ Courant algebroid twisted by $H \in \Omega^4_{cl}(M)$