The Jacobian map, the Jacobian group and the group of automorphisms of the Grassmann algebra

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Plan

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2. The group G of automorphisms of the Grassmann algebra Λ_n and its subgroups.

3. The Jacobian group Σ and the Jacobian map $\mathcal{J}$.

4. The Jacobian ascents.

MOTIVATION

K is a field of characteristic 0

$P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra

$\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n} \in \mathrm{Der}_K(P_n)$

$\sigma \in \mathrm{End}_{K-alg}(P_n)$, $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j})$, the **Jacobian** of σ

$\sigma \mapsto \mathcal{J}(\sigma)$, the **Jacobian map**

The chain rule: $\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))$

The **Jacobian monoid**

$\Sigma := \{\sigma \in \mathrm{End}_{K-alg}(P_n) \,|\, \mathcal{J}(\sigma) = 1, \sigma(x) = x + \cdots\}$

The Jacobian Conj: $\sigma \in \Sigma \Rightarrow \sigma$ is an automorphism, i.e. the Jacobian monoid Σ is a **group**

If JC is true, $K = \overline{K}$, then $\mathrm{Aut}_K(P_n) = \mathrm{Aff}_n(P_n) \times_{ex} \Sigma$, the exact product of groups

Motivation: For the Grassmann algebra Λ_n the Jacobian monoid Σ turns out to be a group, the Jacobian group

The Grassmann algebras and skew derivations

Throughout, K is a *reduced commutative* ring with $\frac{1}{2} \in K$

The *Grassmann alg* (the *exterior alg*) $\Lambda_n = K\lfloor x_1, \dots, x_n \rfloor$:

$$x_1^2 = \cdots = x_n^2 = 0, \;\; x_i x_j = -x_j x_i, \;\; i \neq j.$$

$\mathcal{B}_n = \{\alpha = (\alpha_1, \dots, \alpha_n) \,|\, \alpha_i \in \{0, 1\}\} \simeq \{0, 1\}^n$

$\Lambda_n = \oplus_{\alpha \in \mathcal{B}_n} K x^\alpha$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\mathbb{Z}$-grading: $\Lambda_n = \oplus_{i=0}^n \Lambda_{n,i}$, $\Lambda_{n,i} = \oplus_{|\alpha|=i} K x^\alpha$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$\mathfrak{m} = (x_1, \ldots, x_n)$, $\Lambda_n/\mathfrak{m} \simeq K$, $\mathfrak{m}^{n+1} = 0$

$\mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$-grading: $\Lambda_n = \oplus_{t\in\mathbb{Z}_s}\Lambda_{n,t}$, $\Lambda_{n,t} = \oplus_{i\equiv t \bmod s}\Lambda_{n,i}$

$\mathbb{Z}_2$-grading: $\Lambda_n = \Lambda_n^{ev} \oplus \Lambda_n^{od}$

Exp. fact: skew derivations rather than derivations are more important in studying the Grassmann algebras

A *skew derivation* $\delta : \Lambda_n \to \Lambda_n$ is a K-lin map $\delta(a_ia_j) = \delta(a_i)a_j + (-1)^i a_i\delta(a_j)$, $a_i \in \Lambda_{n,i}$

$\mathrm{SDer}_K(\Lambda_n) \supseteq \oplus_{i=1}^{n}\Lambda_n^{ev}\partial_i$, $\partial_i := \frac{\partial}{\partial x_i}$

$\partial_i(x_1\cdots x_i\cdots x_k) = (-1)^{i-1}x_1\cdots x_{i-1}x_{i+1}\cdots x_k$

$\partial_1^2 = \cdots = \partial_n^2 = 0$, $\partial_i\partial_j = -\partial_j\partial_i$, $i \neq j$. $K\langle\partial_1, \ldots, \partial_n\rangle \simeq \Lambda_n$

$\mathrm{End}_K(\Lambda_n) = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$:

$$x_i^2 = 0, \qquad x_i x_j = -x_j x_i,$$
$$\partial_i^2 = 0, \qquad \partial_i \partial_j = -\partial_j \partial_i,$$
$$\partial_i x_j + x_j \partial_i = \delta_{ij}, \text{ the Kronecker delta.}$$

Elem. $a = a(x_1, \ldots, x_n) = \sum a_\alpha x^\alpha \in \Lambda_n$ should be seen as a polynomial function in anti-comm variables, $a(0) := a_0 \in K = \Lambda_m/\mathfrak{m}$

The Taylor formula: $a = \sum_{\alpha \in \mathcal{B}_n} \partial^\alpha(a)(0) x^\alpha$, $\partial^\alpha := \partial_n^{\alpha_n} \cdots \partial_1^{\alpha_1}$

The group $\mathrm{Aut}_K(\Lambda_n)$ and its subgroups

$G := \mathrm{Aut}_K(\Lambda_n)$, an algebraic group over K

$\mathrm{GL}_n(K)^{op} := \{\sigma_A \,|\, \sigma_A(x_i) = \sum_{j=1}^n a_{ij}x_j,\ A = (a_{ij}) \in \mathrm{GL}_n(K)\}$, $\sigma_A\sigma_B = \sigma_{BA}$

$\mathrm{Inn}(\Lambda_n) := \{\omega_u : x \mapsto uxu^{-1}\}$, the group of inner automorphisms of Λ_n

$\Gamma := \{\gamma_b \,|\, \gamma_b(x_i) = x_i + b_i,\ b_i \in \Lambda_n^{od} \cap \mathfrak{m}^3, i = 1, \ldots, n\}$, $b = (b_1, \ldots, b_n)$

If $K = \mathbb{C}$ the group $\Gamma\mathrm{GL}_n(\mathbb{C})^{op}$ was considered by F. Berezin (1967, Mat Zametki). If $K = k$ is a field of char $\neq 2$ it was proved by D. Djokovic (1978, Canad J Math) that $G = \mathrm{Inn}(\Lambda_k(k)) \rtimes G_{\mathbb{Z}_2-gr}$.

$\Omega := \{\omega_{1+a} \,|\, a \in \Lambda_n^{od}\}$, $\omega_{1+a}\omega_{1+b} = \omega_{1+a+b}$

$U := \{\sigma \in G \,|\, \sigma(x_i) = x_i + \cdots$ for all $i\}$ where the three dots mean bigger terms with respect to the $\mathbb{Z}$-grading

Theorem 1. *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then*

1. $G = U \rtimes \mathrm{GL}_n(K)^{op} = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$

2. $G = \Omega \rtimes G_{\mathbb{Z}_2-gr}$, $G_{\mathbb{Z}_2-gr} = \Gamma \rtimes \mathrm{GL}_n(K)^{op}$

3. $U = \Omega \rtimes \Gamma$ *and* Ω *is a maximal abelian subgroup of* U *if* n *is even* ($\Omega \supseteq U^n$)*; and* $\Omega U^n = \Omega \times U^n$ *is a maximal abelian subgroup of* U *if* n *is odd* ($\Omega \cap U^n = \{e\}$) *where* $U^n := \{\tau_\lambda \,|\, \tau_\lambda(x_i) = x_i + \lambda_i x_1 \cdots x_n,\ \lambda = (\lambda_1, \ldots, \lambda_n) \in K^n\} \simeq K^n$, $\tau_\lambda \leftrightarrow \lambda$

4. $\mathrm{Inn}(\Lambda_n) = \Omega$ *and* $\mathrm{Out}(\Lambda_n) \simeq G_{\mathbb{Z}_2-gr}$

5. $G = G^{ev}G^{od} = G^{od}G^{ev}$ *where* $G^{od} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{od}$ for all $i\}$ *and* $G^{ev} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{ev}$ for all $i\}$

6. $G^{od} = G_{\mathbb{Z}_2-gr}$

7. *Let* $s = 2, \dots, n$. *Then*

(a) If s *is even then* $G_{\mathbb{Z}_s-gr} = \Gamma(s) \rtimes \mathrm{GL}_n(K)^{op}$ *where* $\Gamma(s) := \{\gamma_b \,|\, \text{all } b_i \in \sum_{j\geq 1} \Lambda_{n,1+js}\}$ *and* $\gamma_b(x_i) = x_i + b_i$.

(b) If s *is odd then* $G_{\mathbb{Z}_s-gr} = \Omega(s) \rtimes \mathrm{GL}_n(K)^{op}$ *where* $\Omega(s) := \{\omega_{1+a} \,|\, a \in \sum_{1\leq j \text{ is odd}} \Lambda_{n,js}\}$.

The unique presentation $\sigma = \omega_{1+a}\gamma_b\sigma_A$ for $\sigma \in G$

Each $\sigma \in G = (\Omega \ltimes \Gamma) \ltimes \mathrm{GL}_n(K)^{op}$ is a *unique* product

$$\sigma = \omega_{1+a}\gamma_b\sigma_A$$

$\omega_{1+a} \in \Omega$ ($a \in \Lambda_n'^{od}$), $\gamma_b \in \Gamma$, and $\sigma_A \in \mathrm{GL}_n(K)^{op}$ where $\Lambda_n'^{od} := \oplus_i \Lambda_{n,i}$ and i runs through *odd* natural numbers such $1 \leq i \leq n-1$. The next theorem determines explicitly the elements a, b, and A via the vector-column

$$\sigma(x) := (\sigma(x_1), \ldots, \sigma(x_n))^t$$

(for, only one needs to know explicitly the inverse γ_b^{-1} for each $\gamma_b \in \Gamma$ which is given by the inversion formula below, Theorem 3.

$\Lambda_n = \Lambda_n^{ev} \oplus \Lambda_n^{od} \ni u = u^{ev} + u^{od}$.

Theorem 2 *Each element $\sigma \in G$ is a unique product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ where $a \in \Lambda_n'^{od}$ and*

1. $\sigma(x) = Ax + \cdots$ for some $A \in \mathrm{GL}_n(K)$,

2. $b = A^{-1}\sigma(x)^{od} - x$, *and*

3.

$$a = -\frac{1}{2}\gamma_b(\sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(a'_{i+1}) + \partial_1(a'_1))$$

where $a'_i := (A^{-1}\gamma_b^{-1}(\sigma(x)^{ev}))_i$, the i'th component of the column-vector $A^{-1}\gamma_b^{-1}(\sigma(x)^{ev})$.

The inversion formula σ^{-1} for $\sigma \in G$

Theorem 3. *Let K be a commutative ring, $\sigma \in \Gamma \rtimes \mathrm{GL}_n(K)^{op}$ and $a \in \Lambda_n(K)$. Then*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathcal{B}_n} \lambda_\alpha x^\alpha$$

where

$$\begin{aligned} \lambda_\alpha &:= (1-\sigma(x_n)\partial'_n)\cdots(1-\sigma(x_1)\partial'_1)\partial'^{\alpha}(a) \in K, \\ \partial'^{\alpha} &:= \partial'^{\alpha_n}_n \partial'^{\alpha_{n-1}}_{n-1}\cdots\partial'^{\alpha_1}_1, \end{aligned}$$

$$\partial'_i := \frac{1}{\det(\frac{\partial\sigma(x_\nu)}{\partial x_\mu})} \det \begin{pmatrix} \frac{\partial\sigma(x_1)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_1)}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial\sigma(x_n)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_n)}{\partial x_m} \end{pmatrix},$$

$i = 1, \ldots, n.$

Any $\tau \in G = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$ is a unique product $\tau = \omega_{1+a}\sigma$. Then $\tau^{-1} = \sigma^{-1}\omega_{1-a}$ since $\omega_{1+a}^{-1} = \omega_{1-a}$.

The Jacobian group Σ

$$\Gamma := \{\gamma_b \,|\, \gamma_b(x_i) = b_i = x_i + b'_i,\ b'_i \in \Lambda_n^{od} \cap \mathfrak{m}^3,\ i = 1, \ldots, n\},$$

where the element $b = (b_1, \ldots, b_n)$ should be seen as a vector-function in anti-commuting variables $x_1, \ldots, x_n$, i.e. $b = b(x) = b(x_1, \ldots, x_n)$.

$$\gamma_b \gamma_c = \gamma_{c \circ b}$$

where $c \circ b$ is the composition of functions; namely, the i'th coordinate $(c \circ b)_i$ of the n-tuple $c \circ b$ is equal to $c_i(b_1, \ldots, b_n)$ where $c_i = c_i(x_1, \ldots, x_n)$ (we have substituted elements b_i for x_i in the function $c_i = c_i(x_1, \ldots, x_n)$).

Each $\sigma \in G = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$ is the *unique* product

$$\sigma = \omega_{1+a} \gamma_b \sigma_A,\ \ \omega_{1+a} \in \Omega,\ \gamma_b \in \Gamma,\ \sigma_A \in \mathrm{GL}_n(K)^{op}$$

Let $\sigma' = \omega_{1+a'}\gamma_{b'}\sigma_{A'}$. Then

$$\sigma\sigma' = \omega_{1+a+\gamma_b\sigma_A(a')}\,\gamma_{A^{-1}\sigma_A(b')\circ b}\,\sigma_{A'A}$$

where $\sigma_A(b') := (\sigma_A(b'_1), \ldots, \sigma_A(b'_n))^t$ and $\sigma_A(b')\circ b := (\sigma_A(b'_1) \circ b, \ldots, \sigma_A(b'_n) \circ b)$. This formula shows that the most sophisticated part of the group G is the group Γ.

The even algebra $\Lambda_n^{ev} = \oplus_{m\geq 0}\Lambda_{n,2m}$ belongs to the centre of Λ_n, $E_n := K^* + \sum_{m\geq 1}\Lambda_{n,2m}$ is the group of units of Λ_n^{ev}, $E_n = K^* \times E'_n$ where $E'_n := 1+\sum_{m\geq 1}\Lambda_{n,2m}$, K^* is the group of units of K.

For $\sigma \in \Gamma$, $\frac{\partial\sigma}{\partial x} := (\frac{\partial\sigma(x_i)}{\partial x_j})$ is the **Jacobian matrix** of σ

The chain rule: $\frac{\partial(\sigma\tau)}{\partial x} = \sigma(\frac{\partial\tau}{\partial x}) \cdot \frac{\partial\sigma}{\partial x}$

The **Jacobian map**

$$\mathcal{J} : \Gamma \to E_n', \;\; \sigma \mapsto \mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j}).$$

Theorem 4. *The Jacobian map* $\mathcal{J} : \Gamma \to E_n'$, $\sigma \mapsto \mathcal{J}(\sigma)$, *is surjective if* n *is odd, and it is not surjective if* n *is even but in this case its image is a closed affine subvariety of* E_n' *of codimension* 1 *which is given by a single equation.*

$\mathcal{E}^{od} := \{\sigma \in \text{End}_{K-alg}(\Lambda_n) | \sigma(x_i) - x_i \in \Lambda_n^{od} \cap \mathfrak{m}^3\}$, $\Gamma \subset \mathcal{E}^{od}$.

$$\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau)), \;\; \sigma, \tau \in \mathcal{E}^{od}.$$

Theorem 5. *The monoid*

$$\Sigma = \{\sigma \in \mathcal{E}^{od} \,|\, \mathcal{J}(\sigma) = 1\}$$

is a group, the **Jacobian group**, $\Sigma \subseteq \Gamma$.

Σ is trivial iff $n \leq 3$. So, we always assume that $n \geq 4$. Despite the fact that $\mathcal{J}$ is not a group homomorphism Σ should be seen as the 'kernel' of $\mathcal{J}$.

Theorem 6. *The Jacobian group Σ is not a normal subgroup of Γ iff $n \geq 5$.*

An algebraic group A over K is called **affine** if its algebra of regular functions is a polynomial algebra $K[t_1, \ldots, t_d]$ with coefficients in K where $d := \dim(A)$ is called the **dimension** of A (i.e. A is an *affine space*). If K is a field then $\dim(A)$ is the usual dimension of the algebraic group A over the field K. Below all algebraic groups and varieties are affine.

- **Theorem 7**. *The Jacobian group Σ is an affine group over K of dimension*

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

A subgroup of an algebraic group A over K is called a 1-*parameter subgroup* if it is isomorphic to the algebraic group $(K,+)$. *A minimal set of generators* for an affine group A over K is a set of 1-parameter subgroups that generate the group A as an abstract group but each smaller subset does not generate A.

- *Coordinate functions on* Σ *and a minimal set of generators for* Σ *are given explicitly.*

$\sigma_{i,\lambda x_j x_k x_l} \in \Gamma$: $x_i \mapsto x_i + \lambda x_j x_k x_l$, $x_m \mapsto x_m$, for all $m \neq i$, $\lambda \in K$.

Theorem 8. $\Gamma = \langle \sigma_{i,\lambda x_j x_k x_l} \,|\, i = 1, \dots, n; \lambda \in K; j < k < l \rangle$. *The subgroups* $\{\sigma_{i,\lambda x_j x_k x_l}\}_{\lambda \in K} \simeq (K,+)$ *form a minimal set of generators for* Γ.

The subgroups Σ' and Σ'' of Σ

$\Phi := \{\sigma : x_i \mapsto x_i(1 + a_i) \,|\, a_i \in \Lambda_n^{ev} \cap \mathfrak{m}^2, i = 1, \ldots, n\}$.

To prove the (above) results for Σ we first study in detail two of its subgroups Σ' and Σ'':

$$\Sigma' := \Sigma \cap \Phi = \{\sigma : x_i \mapsto x_i(1 + a_i) \,|\, \mathcal{J}(\sigma) = 1, \\ a_i \in \Lambda_n^{ev} \cap \mathfrak{m}^2,\ 1 \leq i \leq n\}$$

and the subgroup Σ'' is generated by the explicit automorphisms of Σ:

$$\xi_{i,b_i} : x_i \mapsto x_i + b_i, \ \ x_j \mapsto x_j, \ \ j \neq i,$$

where $b_i \in K\lfloor x_1, \ldots, \widehat{x}_i, \ldots, x_n \rfloor^{od}_{\geq 3}$ and $i = 1, \ldots, n$.

The importance of these subgroups is demonstrated by the following two facts.

Theorem 9. $\Sigma = \Sigma'\Sigma''$.

Theorem 10. $\Gamma = \Phi\Sigma''$.

Note that each element x_i is a *normal* element of Λ_n: $x_i\Lambda_n = \Lambda_n x_i$. Therefore, the ideal (x_i) of Λ_n generated by the element x_i determines a coordinate 'hyperplane.' The groups Σ' and Σ'' have the following geometric interpretation: the group Σ' preserves the coordinate 'hyperplanes' and elements of the group Σ'' can be seen as 'rotations.'

By the definition, the group Σ' is a closed subgroup of Σ, it is not a normal subgroup of Σ unless $n \leq 5$. It is not obvious from the outset whether the subgroup Σ'' is closed or normal. In fact, it is.

Theorem 11. *Σ'' is the closed normal subgroup of Σ, Σ'' is an affine group of dimension*

$$\dim(\Sigma'') = \begin{cases} (n-1)2^{n-1} - n^2 + 2 - (n-3)\binom{n}{2} \\ (n-1)2^{n-1} - n^2 + 1 - (n-3)\binom{n}{2} \end{cases}$$

where n is even and odd resp., and the factor group Σ/Σ'' is an abelian affine group of dimension $(n-3)\binom{n}{2}$.

Theorem 12. *The group Σ' is an affine group over K of dimension*

$$\dim(\Sigma') = \begin{cases} (n-2)2^{n-2} - n + 2 & \text{if } n \text{ is even,} \\ (n-2)2^{n-2} - n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 13. *The intersection $\Sigma' \cap \Sigma''$ is a closed subgroup of Σ, it is an affine group over K of dimension*

$$\dim(\Sigma' \cap \Sigma'') = \begin{cases} (n-2)2^{n-2} - n + 2 - (n-3)\binom{n}{2} \\ (n-2)2^{n-2} - n + 1 - (n-3)\binom{n}{2} \end{cases}$$

where n is even and odd resp.

- *The coordinates on Σ' and Σ'' are given explicitly.*

To find coordinates for the groups Σ, Σ', and Σ'' explicitly, we introduce avoidance functions and a series of subgroups $\{\Phi'^{2s+1}\}$, $s = 1, 2, \ldots, [\frac{n-1}{2}$ of Φ that are given explicitly. They are too technical to explain.

The Jacobian ascents Γ_{2s}

In order to study the image of the Jacobian map $\mathcal{J} : \Gamma \to E'_n$, $\sigma \mapsto \mathcal{J}(\sigma)$, certain overgroups of the Jacobian group Σ are introduced. They are called the *Jacobian ascents*. The problem of finding the image $\mathrm{im}(\mathcal{J})$ is equal to the problem of finding generators for these groups. Let us give some details.

The Grassmann algebra Λ_n has the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^i\}$. Therefore, the group E'_n has the induced $\mathfrak{m}$-adic filtration:

$$\begin{aligned} E'_n &= E'_{n,2} \supset \cdots \supset E'_{n,2m} \supset \cdots \supset E'_{n,2[\frac{n}{2}]} \\ &\supset E'_{n,2[\frac{n}{2}]+2} = \{1\}, \end{aligned}$$

where $E'_{n,2m} := E'_n \cap (1 + \mathfrak{m}^{2m})$. Correspondingly, the group Γ has the **Jacobian filtration**:

$$\begin{aligned} \Gamma &= \Gamma_2 \supseteq \Gamma_4 \supseteq \cdots \supseteq \Gamma_{2m} \\ &\supseteq \cdots \supseteq \Gamma_{2[\frac{n}{2}]} \supseteq \Gamma_{2[\frac{n}{2}]+2} = \Sigma, \end{aligned}$$

where

$$\Gamma_{2m} := \mathcal{J}^{-1}(E'_{n,2m}) = \{\sigma \in \Gamma \mid \mathcal{J}(\sigma) \in E'_{n,2m}\}.$$

It follows from the equality $\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma)\sigma(\mathcal{J}(\tau))$ that all Γ_{2m} are subgroups of Γ, they are called, the **Jacobian ascents** of the Jacobian group Σ.

The Jacobian ascents are *distinct* groups with a *single* exception when two groups coincide.

This is a subtle fact, it explains (partly) why formulae for various dimensions differ by 1 in odd and even cases.

Theorem 14 *Let K be a commutative ring and $n \geq 4$.*

(a) *If n is an odd number then the Jacobian ascents*

$$\Gamma = \Gamma_2 \supset \cdots \supset \Gamma_{2s} \supset \cdots \supset \Gamma_{2[\frac{n}{2}]} \supset \Gamma_{2[\frac{n}{2}]+2} = \Sigma$$

are distinct groups.

(b) *If n is an even number then the Jacobian ascents*

$$\Gamma = \Gamma_2 \supset \cdots\cdots \supset \Gamma_{2[\frac{n}{2}]-2} \supset \Gamma_{2[\frac{n}{2}]} = \Gamma_{2[\frac{n}{2}]+2} = \Sigma$$

are distinct groups except the last two groups, i.e. $\Gamma_{2[\frac{n}{2}]} = \Gamma_{2[\frac{n}{2}]+2}$.

The subgroups $\{\Gamma^{2s+1}\}$ of Γ are given explicitly,

$$\Gamma^{2s+1} := \{\sigma : x_i \mapsto x_i + a_i \,|\, a_i \in \Lambda_n^{od} \cap \mathfrak{m}^{2s+1}, \\ 1 \leq i \leq n\}, \;\; s \geq 1,$$

they have clear structure. The next result explains that the Jacobian ascents $\{\Gamma_{2s}\}$ have clear structure too, $\Gamma_{2s} = \Gamma^{2s+1}\Sigma$, and so the structure of the Jacobian ascents is completely determined by the structure of the Jacobian group Σ.

Theorem 15. *Let K be a commutative ring and $n \geq 4$. Then $\Gamma_{2s} = \Gamma^{2s+1}\Sigma$ for each* $s = 1, 2, \ldots, [\frac{n-1}{2}]$.