Covers of Elliptic Curves and Excellence

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Exponential Maps

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1$$

Let E be a complex elliptic curve.

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{C} \xrightarrow{\exp} E(\mathbb{C}) \longrightarrow 0$$

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Assume

- E is over a number field k_0
- $\mathrm{End}(E) \cong \mathbb{Z}$.

Language and first order theory T_E:

- Two sorts: $V \xrightarrow{\exp} E$.
- $V = \langle V; +, (q\cdot)_{q\in\mathbb{Q}} \rangle$, $\mathbb{Q}$-vector space.
- On E: relation for each k_0-Zariski-closed subset of E^n; $\emptyset$-bi-interpretable with ACF^{k_0}.
- $exp : \langle V; + \rangle \xrightarrow{\exp} \langle E; + \rangle$ surjective homomorphism.

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Add constants ω_1, ω_2 and the $L_{\omega_1,\omega}$ axiom:

$$(\omega_1, \omega_2) \text{ is a } \mathbb{Z}\text{-basis of } \mathrm{ker}(\mathrm{exp}). \qquad (\mathrm{ker} \cong \mathbb{Z}^2)$$

Let $T_E^{\mathbb{Z}^2} := T_E \cup \{(\mathrm{ker} \cong \mathbb{Z}^2)\}$.

Remark

If $A \leq \mathcal{M} \models T_E^{\mathbb{Z}^2}$ and $A = \langle V(A) \rangle^{\mathcal{M}}$

- $\mathrm{ker} \leq V(A) \leq_{div} V(\mathcal{M})$
- $\mathrm{Tor}(E) \leq E(A) = \exp(V(A)) \leq_{div} E(\mathcal{M})$.

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Categoricity

Theorem (B, Gavrilovich, Zilber)

Suppose $\mathcal{M}_1, \mathcal{M}_2 \models T_E^{\mathbb{Z}^2}$, *and suppose*

- $\langle \emptyset \rangle^{\mathcal{M}_1} \cong \langle \emptyset \rangle^{\mathcal{M}_2}$
- $\dim(E(\mathcal{M}_1)) = \dim(E(\mathcal{M}_2))$

Then $\mathcal{M}_1 \cong \mathcal{M}_2$.

Remark

Only finitely many isomorphism types for $\langle \emptyset \rangle^{\mathcal{M}}$.

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Independent Systems

Let $\mathcal{M} \models T_E^{\mathbb{Z}^2}$.

- Field-theoretic acl induces a closure operator cl
- $\mathrm{cl}^{\mathcal{M}}(A) \preceq \mathcal{M}$
- $\mathrm{cl}^{\mathcal{M}}$ induces a pregeometry on $V(\mathcal{M})$.

Definition

Let $N := \{0, \ldots, N-1\} \geq 0$. A system $(\mathcal{M}_s)_{s \subseteq N}$ of submodels of a model $\mathcal{M} \models T_E^{\mathbb{Z}^2}$ is an *independent system* iff

- $\mathcal{M}_s \underset{\mathcal{M}_{s\cap t}}{\smile\!\!\!\!\!\mid\;} \mathcal{M}_t$
- $\mathcal{M}_s = \mathrm{cl}^{\mathcal{M}}(\bigcup_{i \in s} \mathcal{M}_i)$

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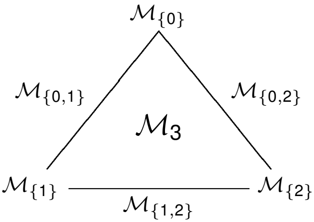

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Thumbtack Lemma

Theorem (Thumbtack Lemma)

- $(\mathcal{M}_s)_{s\subseteq N}$ *an independent system of submodels of* $\mathcal{M} \models T_E^{\mathbb{Z}^2}$, $N \geq 0$
- $H := \left\langle \bigcup_{s\subsetneq N} \mathcal{M}_s \right\rangle^{\mathcal{M}}$
- $H \leq_{\text{fg}} A \leq \mathcal{M}$

Then there exists a basis $\overline{a}$ *of* A *over* H *such that*

$$\text{qftp}(\exp(\overline{a})/\exp(H)) \models \text{qftp}(\overline{a}/H).$$

$$\text{qftp}((\exp(\overline{a}/n))_{n\in\mathbb{N}}/\exp(H)) \models \text{qftp}(\overline{a}/H),$$

so the condition on $\overline{a}$ is equivalent to: for each m, all m-division points of $\exp(\overline{a})$ are field-conjugate over $k_0(\exp(H), \exp(\overline{a}))$.

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Then there exists a basis $\bar{a}$ *of* A *over* H *such that*

$$\operatorname{qftp}(\exp(\bar{a}) / \exp(H)) \models \operatorname{qftp}(\bar{a}/H).$$

$$\operatorname{qftp}((\exp({}^{\bar{a}}/_{n}))_{n \in \mathbb{N}} / \exp(H)) \models \operatorname{qftp}(\bar{a}/H),$$

so the condition on $\bar{a}$ is equivalent to: for each m, all m-division points of $\exp(\bar{a})$ are field-conjugate over $k_0(\exp(H), \exp(\bar{a}))$.

Corollary of Thumbtacks

Corollary

- $(\mathcal{M}_s)_{s \subseteq N}$ *an independent system of* countable *submodels of* $\mathcal{M} \models T_E^{\mathbb{Z}^2}$, $N \geq 0$
- $H := \left\langle \bigcup_{s \subsetneq N} \mathcal{M}_s \right\rangle^{\mathcal{M}}$
- $H \leq_{\mathrm{fg}} A \leq \mathcal{M}$

Then any embedding

$$\sigma : A \hookrightarrow \mathcal{M}' \models T_E^{\mathbb{Z}^2}$$

extends to an isomorphism

$$\mathrm{cl}^{\mathcal{M}}(A) \xrightarrow{\cong} \mathrm{cl}^{\mathcal{M}'}(\sigma A) .$$

Proof of Theorem 1

Suppose $\mathcal{M}_1, \mathcal{M}_2 \models T_E^{\mathbb{Z}^2}$, and suppose

- $\langle\emptyset\rangle^{\mathcal{M}_1} \cong \langle\emptyset\rangle^{\mathcal{M}_2}$
- $\dim(\mathcal{M}_1) = \dim(\mathcal{M}_2)$

Let $(\alpha_i^j)_{i\in I}$ enumerate a cl-basis of $\mathcal{M}_j$.
For $s \subseteq I$, let $A_s^j := \mathrm{cl}^{\mathcal{M}_j}((\alpha_i^j)_{i\in s})$.
By the Corollary ($N = 0, A = H = A_\emptyset^1$), exists

$$\sigma_\emptyset : A_\emptyset^1 \xrightarrow{\cong} A_\emptyset^2 \ .$$

By the Corollary ($N = 1, A = \langle A_\emptyset^1, \alpha_i^1 \rangle^{\mathcal{M}_1}$), for $i \in I$ exists

$$\sigma_{\{i\}} : A_{\{i\}}^1 \xrightarrow{\cong} A_{\{i\}}^2$$

extending $\sigma_\emptyset$ such that $\sigma_{\{i\}}(\alpha_i^1) = \alpha_i^2$.

Proof of Theorem 1 *cont*^{d}

Now suppose $s \subset_{\text{fin}} I$ and for $t \subsetneq s$ we have $\sigma_t : A^1_t \xrightarrow{\cong} A^2_t$, agreeing on intersections. Then by N-uniqueness for ACF_0,

$$\left\langle \bigcup_{t \subsetneq s} \sigma_t \right\rangle : \langle (A^1_t)_{t \subsetneq s} \rangle^{\mathcal{M}_1} \xrightarrow{\cong} \langle (A^2_t)_{t \subsetneq s} \rangle^{\mathcal{M}_2} ,$$

which by the Corollary ($N = |s|$, $A = H = \langle (A^1_t)_{t \subsetneq s} \rangle^{\mathcal{M}_1}$) extends to

$$\sigma_s : A^1_s \xrightarrow{\cong} A^2_s .$$

So

$$\sigma := \bigcup_{s \subset_{\text{fin}} I} \sigma_s : \mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}_2 .$$

Thumbtack Lemma Again

Lemma (Thumbtack Lemma)

- $(\mathcal{M}_s)_{s \subseteq N}$ *an independent system of submodels of* $\mathcal{M} \models T_E^{\mathbb{Z}^2}$, $N \geq 0$
- $H := \left\langle \bigcup_{s \subsetneq N} \mathcal{M}_s \right\rangle^{\mathcal{M}}$
- $H \leq_{\mathrm{fg}} A \leq \mathcal{M}$

Then there exists a basis $\overline{a}$ of A over H such that

$$\mathrm{qftp}(\exp(\overline{a}) / \exp(H)) \models \mathrm{qftp}(\overline{a}/H).$$

The condition on $\overline{a}$ is equivalent to: for each m, all m-division points of $\exp(\overline{a})$ are field-conjugate over $k := k_0(\exp(H), \exp(\overline{a}))$. In particular, the subgroup generated by $\exp(\overline{a})$ is pure in $E(k)$.

Thumbtack Lemma Again

Lemma (Thumbtack Lemma)

- *$(\mathcal{M}_s)_{s \subseteq N}$ an independent system of submodels of $\mathcal{M} \models T_E^{\mathbb{Z}^2}$, $N \geq 0$*
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Proof of the Thumbtack Lemma

Theorem

Let k be a finitely generated extension of $k_0(\exp(H))$. Then

$$E(k)/_{\exp(H)}$$

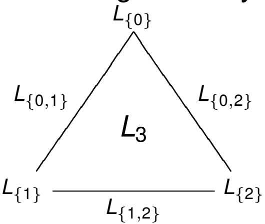
is locally free.

Remark

- Locally free means: pure hull of any finitely generated subgroup is finitely generated.
- The Thumbtack Lemma then follows via Kummer theory.

Sketch Proof of ${}^{E(k)}/_{G}$ locally free

Let $G := \exp(H)$. Proceed by induction on N. $N = 1$: By Lang-Néron, ${}^{E(k)}/_{G}$ is even finitely generated.
Consider case $N = 3$. We have the independent system of algebraically closed fields:



and $k = L_{\{0,1\}} L_{\{1,2\}} L_{\{0,2\}}(\overline{\beta})$ say. We may assume $\overline{\beta} \in L_3$.
Let $\overline{b} \in E(k)^n$.

Lemma

There exists $k_1 \geq_{fin} L_{\{0,1\}} L_{\{0,2\}}(\overline{\beta}, \overline{b})$ and a place $\pi : L_3 \to_{L_{\{1,2\}}} L_{\{1,2\}}$ such that

- $\pi k_1 \subseteq k_1$
- $\pi(L_{\{0,1\}} L_{\{0,2\}}) = L_{\{1\}} L_{\{2\}}$

Sketch Proof of ${}^{E(k)}/_{G}$ locally free *cont*^{d}

Lemma

$\mathrm{pureHull}_{E(k)}(E(k_1)) = \mathrm{pureHull}_{E(k_1)+E(L_{\{1,2\}})}(E(k_1))$.

$$\mathrm{pureHull}_{E(k)/G}\left(\left\langle \overline{b}/G \right\rangle\right) = {}^{\mathrm{pureHull}_{E(k)}(\langle \overline{b}\rangle)}/_{G}$$
$$= {}^{\mathrm{pureHull}_{E(k_1)+E(L_{\{1,2\}})}(\langle \overline{b}\rangle)}/_{G}$$
$$\leq {}^{\mathrm{pureHull}_{E(k_1)}(\langle \overline{b},\pi(\overline{b})\rangle)}/_{G}$$

(since if $m(\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) \in \langle \overline{b} \rangle$, then $\gamma := (\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) - \pi(\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) = \alpha_{k_1} - \pi\alpha_{k_1} \in \mathrm{pureHull}_{E(k_1)}(\langle \overline{b}, \pi\overline{b}\rangle)$, and $\gamma = \alpha_{k_1} + \alpha_{L_{\{1,2\}}}$ mod G.
So subgroup of quotient of ${}^{\mathrm{pureHull}_{E(k_1)}(\langle \overline{b},\pi\overline{b}\rangle)}/_{E(L_{\{1\}})+E(L_{\{2\}})}$, which is f.g. by induction, so f.g.

Theorem

Let k be a finitely generated extension of $k_0(\exp(H))$; suppose $\overline{a}$ is simple in $E(k)$.
Then the left image of the $\bar{k}/k$-Kummer-Tate pairing,

$$Z := \langle \mathrm{Gal}(\bar{k}/k), \overline{a} \rangle_{\infty}^{\bar{k}/k} \leq T^n,$$

is of finite index in T^n.

Where $T \cong \hat{\mathbb{Z}}^2$ is the product of the Tate modules, and

$$\begin{array}{cccccc} \langle \cdot, \cdot \rangle_{\infty}^{\bar{k}/k} & : & \mathrm{Gal}(\bar{k}/k) \times E(k) & \to & T \\ & ; & (\sigma, a) & \mapsto & (\sigma(\frac{1}{n}a) - \frac{1}{n}a)_n. \end{array}$$