Defining the integers in expansions of the real field

Philipp Hieronymi

McMaster University
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Let $\overline{\mathbb{R}} = (\mathbb{R}, +, \cdot)$ be the field of real numbers.

Theorem - H.

Let $D \subseteq \mathbb{R}$ be closed and discrete and $f : D^n \to \mathbb{R}$ be such that $f(D^n)$ is somewhere dense. Then $(\overline{\mathbb{R}}, f)$ defines $\mathbb{Z}$.

This is really about being able to do approximation.

Suppose $n = 1$ and $f(D)$ is a dense subset $(1, 2)$. Consider the following definable set:

$$\{x \in (1,2) \ : \ \forall a \in D \exists b \in D \ a < b \wedge f(b) < x < f(b)(1 + b^{-2})\}$$

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Theorem - Friedman, Miller

Let $\mathcal{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ and let $D \subseteq \mathbb{R}$ be such that, for every $m \in \mathbb{N}$ and $f : \mathbb{R}^m \to \mathbb{R}$ definable in $\mathcal{R}$, the image $f(D^n)$ is nowhere dense. Then every subset of $\mathbb{R}$ definable in $(\mathcal{R}, D)$ either has interior or is nowhere dense.

Dichotomy

Let $\mathcal{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ and let $D \subseteq \mathbb{R}$ be closed and discrete. Then either

- $(\mathcal{R}, D)$ defines $\mathbb{Z}$ or
- every subset of $\mathbb{R}$ definable in $(\mathcal{R}, D)$ has interior or is nowhere dense.

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Let $\alpha, \beta \in \mathbb{R}_{>0}$ with $\log_\alpha(\beta) \notin \mathbb{Q}$. Then $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ defines $\mathbb{Z}$.

Proof: The set $\alpha^{\mathbb{N}} \cup \beta^{\mathbb{N}}$ is closed and discrete. Moreover, it is definable in $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ and its set of quotients is dense in $\mathbb{R}_{>0}$.

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Remark

- van den Dries, 85, $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}})$ does not define $\mathbb{Z}$,
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Miller's program

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Let S be an infinite cyclic subgroup of $(\mathbb{C}^{\times}, \cdot)$. Then exactly one of the following holds:

- $(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$,
- $(\overline{\mathbb{R}}, S)$ is d-minimal,
- every open definable set in $(\overline{\mathbb{R}}, S)$ is semialgebraic.

Proof: Let $S := (ae^{i\varphi})^{\mathbb{Z}} \subseteq \mathbb{R}^2$. If $a = 1$, S is a finitely generated subgroup of the unit circle. If $a \neq 1$ and $\varphi \in 2\pi\mathbb{Q}$, then $(\overline{\mathbb{R}}, S)$ and $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$ are interdefinable.

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Finally let $a \neq 1$ and $\varphi \notin 2\pi\mathbb{Q}$. Then the function

$$(a_1, a_2) \mapsto \sqrt{a_1^2 + a_2^2}$$

is injective on S and maps $(ae^{i\varphi})^n$ to a^n for every $n \in \mathbb{Z}$. Further the function

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maps $(ae^{i\varphi})^n$ to $\sin(n\varphi)$ for every $n \in \mathbb{Z}$.

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Miller's Asymptotic Extraction of Groups

An expansion of $\overline{\mathbb{R}}$ defines $\mathbb{Z}$ iff it defines the range of a sequence $(a_k)_{k\in\mathbb{N}}$ of real numbers such that $\lim_{k\to\infty}(a_{k+1} - a_k) \in \mathbb{R} - \{0\}$.

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An expansion of $\overline{\mathbb{R}}$ defines $\mathbb{Z}$ iff it defines the range S of an increasing sequence $(a_k)_{k\in\mathbb{N}}$ of positive real numbers such that S is closed and discrete and $\sup_k(a_{k+1} - a_k) \in \mathbb{R}_{>0}$.

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Proof of the Main Theorem

Let $D \subseteq \mathbb{R}$ be closed and discrete and $f : D^n \to \mathbb{R}$ such that $f(D^n)$ is somewhere dense. We can assume that D is a subset of $\mathbb{R}_{\geq 1}$, n equals 1, $f(D)$ is dense in $(1, 2)$.

Idea

Find an $c \in \mathbb{R}$ and a sequence $(d_N)_{N \in \mathbb{N}}$ of elements such that:

(1) $\{d_N : N \in \mathbb{N}\} = \{d \in D : f(d) < c < f(d)(1 + d^{-2})\}$,

(2) $f(d_N)(1 + \frac{d_N^{-2}}{N + \frac{1}{N}}) < c < f(d_N)(1 + \frac{d_N^{-2}}{N})$.

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In order to make the idea work, one has to add the following in (1)

$$\forall e \in D (d^{\frac{1}{7}} < e < d) \to \neg(f(e) < c < f(e) \cdot (1 + e^{-2})).$$

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