Stabilizers

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T a theory, $\mathbb{U}$ a universal domain. Definable = $\mathbb{U}$-definable. $A \leq \mathbb{U}$ small. A-invariant means: $Aut(\mathbb{U}/A)$-invariant. Suppress A if $A = \emptyset$.

Let D be a definable set (or a countable union of definable sets). An *ideal* I on D is an ideal of the Boolean algebra of $\mathbb{U}$-definable subsets of D.

Definition

An A-invariant ideal I is S1 if whenever a_i is an A-indiscernible sequence, and $\phi(x, a_i) \wedge \phi(x, a_j) \in I$ for $i \neq j$, we have $\phi(x, a_i) \in I$.

Examples of S1 ideals.
$(\phi(x, a_i) \wedge \phi(x, a_j) \in I \implies \phi(x, a_i) \in I.)$

1. The complement I of an invariant global type q.
2. (Finite S1 rank). If $S1(D) = n$, let $I = \{D' : S1(D') < n\}$.
3. The forking ideal $I_{fork/A}$, is contained in any S1-ideal over A. If T is simple, or NIP, it is an S1-ideal.
4. The measure-zero ideal of any $Aut(\mathbb{U})$-invariant, finitely additive $\mathbb{R}$-valued measure on definable subsets of D.
   - Keisler measures, NIP
   - Ultraproducts of measures.
   - A natural S1-ideal on D when D is pseudo-finite.

For the rest of this talk, I denotes an S1-ideal.

A set is *wide* if it is not contained in any definable D with $D \in I$.

Almost all / almost none dichotomy

- An invariant relation $R(x, y)$ is *stable* if for any A and any A-invariant global types $p(x), q(y)$, if $p(x) \otimes q(y)$ implies R then so does $q(y) \otimes p(x)$.
- Let R be stable, and assume an invariant type $q(y)$ exists, extending $tp(b)$. Then $R(a, b)$ holds for *all* or *no* a such that $tp(a/b)$ does not fork over $\emptyset$.
- Let $(D_x), (D'_y)$ be definable families of definable sets. The relation:

$$D_x \cap D'_y \text{ is wide}$$

  is stable.

Independence theorem

- Assume:
  $tp(a)$ extends to an invariant global type,
  $tp(b/a)$ does not divide over $\emptyset$, and
  $tp(c/a,b)$ is wide.
- Let $tp(b) = tp(b')$, s.t. $tp(b'/a)$ does not divide over $\emptyset$.
- Then there exists c' with $tp(c'/a,b')$ wide, and
  $tp(c'b') = tp(cb), tp(c'a) = tp(ca)$.

Theorem (Another version)

Let M be a model. Let μ_x, μ_y, μ_z be commuting measures. Then there exist measure-one subsets Ω_w of S_w for $w \subset \{x,y,z\}$ with $|w| = 2$, with the following amalgamation property. Assume $q_w \in S_w$, $q_w|i = q_{w'}|i$ for $i \in w, w' \subset \{xyz\}$. Then there exists $q \in S_{xyz}(\mu)$, $q|w = q_w$ for $w \subset \{x,y,z\}$.

Let G be a definable group, X a definable subset of G.

X, X' are *comparable* if each one is contained in finitely many right translates of the other. We are interested in comparability classes.

I is a (right) translation invariant, S1- ideal on the group generated by X.

Note in measure setting, this means $\mu(X^{-1}X) \leq k\mu(X)$, k finite; so e.g. cannot have $k+1$ disjoint X-translates of X.

Ideal explanation: X is comparable to a subgroup of G. But,

Example

L be a connected Lie group, X a compact neighborhood of 1. Then the Haar measure μ measures $G = < X >$, but X is not comparable to a subgroup.

Stabilizer theorem

Assume X is wide for some S1-ideal on $XX^{-1}X$.

Theorem

There exist a $\bigvee$-definable $\widetilde{G}$ and an $\bigwedge$-definable $\Gamma \subseteq \widetilde{G}$, such that $\widetilde{G}/\Gamma$ is bounded; and any definable D with $\Gamma \leq D \leq \widetilde{G}$ is comparable to X.

$\widetilde{G}/\Gamma$ admits the structure of a connected, finite-dimensional Lie group. The compact open neighborhoods of L are intertwined with the definable sets containing Γ, contained in $\widetilde{G}$.

$\widetilde{G}, \Gamma$ can be defined without parameters.

Some historical background:

Zilber If X is an irreducible definable subset of G, and $\dim(XX) = \dim(X)$, there exists a definable subgroup H of G such that $X \triangle H$ is small.

CH-QF for quasi-finite dimension, assuming definability. Initially proved CSFG-empirically and inductively. Then using stability of the relation: $\delta(X_a \cap X_b) < \alpha = \delta(X_a) = \delta(X_b)$.

PAC Proof extended to finite S1 dimension, still assuming definability. Part 2 in this setting reads: Γ is an intersection of definable groups.

Kim-Pillay Independence theorem for simple theories, $I = I_{fork}$, without definability; ∞-definable stable relation. (Cf. also Lazy guide.) Pillay, Wagner, supersimple groups.

Lascar Connected topological groups. The "Lie" conclusion uses the Gleason- Yamabe structure theory for locally compact groups: every locally compact group G has an open subgroup G_1 which is isomorphic to a projective limit of Lie groups.

A finite combinatorics - model theory dictionary

$K = \Pi_D K_i$ an ultraproduct.
A definable subset X is *pseudo-finite* if $X(K_i)$ is finite for almost all i.

$$|X| = \Pi_D |X_i| \in \mathbb{R}^*$$

$$\delta(X) = \log |X| \in \mathbb{R}^* / < \mathbb{R} >$$

where $< \mathbb{R} >$ is the convex hull of $\mathbb{R}$ in $\mathbb{R}^*$.
δ has the properties of a dimension theory. Moreover,

$$\mu(Y) = st(|Y|/|X|)$$

is a measure on definable sets Y with $\delta(Y) = \delta(X)$.

- By expanding the language, we can arrange that μ is definable.
- Many two way translations. Example.
- An alternative regime, not discussed here: replace $< \mathbb{R} >$ by $\{r : |r| << |X|\}$. I.e replace $|Y| \leq K|X|$ by the weaker $|Y| \leq |X|^{1+\epsilon}$.

Sum-product phenomenon

Definition

G a group (field). A finite subset X of G is a *k-near-subgroup* if $|XX^{-1}X| \leq k|X|$.

Really, a family X_i of k-near-subgroups is considered.
Translates to:

$$\delta(X) = \delta(XX^{-1}X)$$

So $\mu = \mu_X$ measures $XX^{-1}X$.

A *k-approximate subgroup* is a set X with $X = X^{-1}$ and $XX \subseteq XF$ for some $|F|$ with $|F| \leq k'$. Near-subgroups are contained in a finite union of cosets of an approximate subgroup, of the same size up to a bounded multiple. [Tao-Vu, Additive combinatorics.]

- $[-N, N]$ is an approximate subgroup of $\mathbb{Z}$.
- $[-N, N]^2 \times [-N^2, N^2]$ is an approximate subgroup of the Heisenberg group; in general balls in nilpotent groups are approximate subgroups.
- For Abelian groups G there is a good description of near-subgroups, Freiman-Green-Rusza. Group extensions, Tao.
- "*The open question is to formulate an analogous conjectural classification in the non-abelian setting ... of finite sets A in a multiplicative group G for which* $|A \cdot A| \leq O(1)|A|$ *(Tao).*
  http://terrytao.wordpress.com/2007/03/02/open-question-noncomm
- One expects that in sufficiently non-Abelian groups, or for sufficiently nontrivial rings, approximate subgroups (subrings) are close to subgroups (subrings).

Selected results from combinatorial literature

- Erdoš, P.; Szemerédi, E. On sums and products of integers. Studies in pure mathematics, 213–218, Birkhuser, Basel, 1983. Conjecture: $X \subset \mathbb{Z}^*$ weakly quasi-finite implies $\delta((X+X) \cup XX) = 2\delta(X)$.
- Bourgain, J., Katz, N.H., Tao, T.C.: A sum-product estimate in nite elds and applications, Geom. Funct. Anal. 14 (2004), no. 1, 27-57. No quasi-finite near-subfields of $\mathbb{F}_{p^*}$.
- Helfgott, H. A. Growth and generation in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. Ann. of Math. (2) 167 (2008), no. 2, 601–623. No Zariski dense quasi-finite subgroups of $SL_2(p^*)$. Dinai, for $SL_2(q^*)$. Chang, $SL_3(\mathbb{R})$. Helfgott, $SL_3(\mathbb{F}_p)$. (Strong régime.)
- Helfgott: if $\delta(X) > \epsilon\delta(G)$, $G = SL_2(\mathbb{F}_{p^*})$, then X generates G in boundedly many steps. Compare Zilber's irreducibllity theorem in finite Morley rank.

- Terence Tao, The sum-product phenomenon in arbitrary rings, arXiv:0806.2497
  Let X be near-subring of a field F. Then there exists a subring F' of F of cardinality at most a bounded multiple of $|X|$, a bounded set B, and $a \in F \setminus 0$ such that $X \subseteq aF' + B$. Similar statement for division rings.
- Tao- Balog-Szemerédi-Gowers. , in Tao, Product-set estimates for non-commutative groups.
  Assume $\delta(A) = \delta(B) = \alpha$, and $\delta(E(a,b)) \geq 3\alpha$, where $E(A,B) = \{(a,b,a',b') \in A \times B : ab = a'b'\}$. Then there exists $A' \subset A, B' \subset B$ with $\delta(A') = \delta(B') = \alpha$, and $\delta(A'B') \leq \alpha$.
- Bourgain et al, Gowers, Wigderson, applications to exponential sums, Van den Waerden density bounds, computer science.
- Compare independence theorem (measure version) to Komlos-Simonovitz corollary to Szemerédi.

Theorem (Gromov 81, van den Dries, -Wilkie 84, Kleiner 2009)

Let Γ be a finitely generated group. If all (infinitely many) balls X in the Cayley graph are near-subgroups, then G is nilpotent.

- Linear case follows from Tits' alternative.
- The solvable case, conjectured by Bass-Serre, was proved by Milnor-Woolf.
- All proofs reduce to the linear case. Gromov - and Van den Dries - Wilkie - do so using Montgomery-Zippin.

Conjecture (B. Green)

Let X be a near-subgroup of a group G. A large subset of X is contained in a "Bourgain system" ; equivalently there exist

$$X' \supset X_1 \supset X_2 \supset \cdots$$

$$1 \in X_n^{-1} = X_n, \; X_{n+1}X_{n+1} \subseteq X_n$$

and $|X_n| \leq C|X_{n+1}|$ *with C bounded.*

Remark

- *(I'm not sure how the $\cdots$ is intended.)*
- *Without the C, this states precisely that an ∞-definable stabilizer exists, $\cap_n X_n$.*

Two corollaries of Stabilizer theorem

Theorem

Let $f : \mathbb{N}^2 \to \mathbb{N}$ be any function, and fix $k \in \mathbb{N}$. Then there exist $e, c, N \in \mathbb{N}$ with $N > f(e, c)$ such that the following holds.
Let G be any group, X a finite subset, and assume
$|XX^{-1}X| \leq k|X|$.
Then there are subsets $X_N \subset X_{N-1} \subset \cdots \subset X_1 \subset X^{-1}XX^{-1}X$, such that X is contained in $\leq e$ translates of X_1, and for $1 \leq m, n < N$ we have:

1. $X_n = X_n^{-1}$
2. $X_{n+1}X_{n+1} \subseteq X_n$
3. X_n *is contained in* $\leq c$ *translates of* X_{n+1}.
4. $[X_n, X_m] \subseteq X_k$ *whenever* $k \leq N$ *and* $k < n + m$ *In particular each* X_n *is closed under* $[,]$.
5. $X_{n+1} = \{x : x^4 \in X_n\}$

The proof is by "transfer". (3) comes from the fact that in a Lie group, balls of radius ϵ have volume about c^{ϵ}.

Theorem

Let G be a semisimple linear algebraic group (or a simple group of finite Morley rank) over $K = UltK_i$, $\Gamma = \cap_n Y_n$ a Zariski dense subgroup of G, with Y_n definable, $\delta(Y_n) = \alpha$. Then Γ is definable.

Generalizes Helfgott, Chang,Dinai for $G = SL_2$, SL_3, $K = \mathbb{F}_{q^*}$ or $\mathbb{R}^*$, in "near-subgroup" setting.

Proof:

- Can take Γ normal in $\widetilde{G}$.
- Extend δ to $\bigwedge$-definable setes. Let $\delta_\Gamma(Y) = \delta(Y \cap \Gamma)$. H.-Wagner, following Larsen-Pink:

$$\delta_\Gamma(Y)/\dim(Y) \leq \delta_\Gamma(X)/\dim(X)$$

- Define $f : G^m \to G$, $f(x_1, \dots, x_n) = a_1^{x_1} \cdot \cdots \cdot a_n^{x_n}$. Then

$$\delta(f(\Gamma^m)) = \delta(\Gamma)/\dim(G)$$

- Γ contains a *definable* set of the same dimension, $W = f(X^m) = a_1^X \cdot \cdots \cdot a_n^X$.
- Γ is contained in finitely many translates of $W^{-1}W$.