The fine structure of models of classifiable theories

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Joint work with
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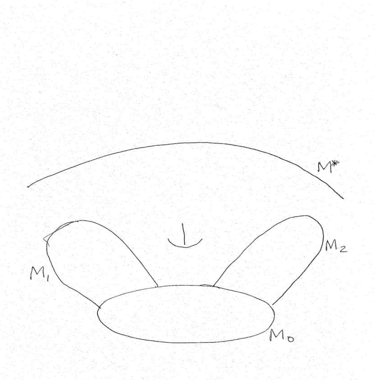
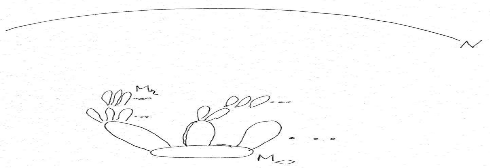
Definition

A complete, countable theory T is classifiable if it is superstable and there is a prime and minimal model M^* over any independent triple (M_0, M_1, M_2) of models.



Theorem (Shelah, Shelah-Buechler)

A countable, complete theory T is classifiable if and only if every $N \models T$ is prime and minimal over an independent tree $\{M_\eta : \eta \in I\}$ of countable na-substructures.



$M \subseteq_{na} N$ means $M \preceq N$ and for all M-definable D and all finite $F \subseteq M$, if $D^N \setminus \mathrm{acl}(F)$ is nonempty, then so is $D^M \setminus \mathrm{acl}(F)$.

Theorem (Hart,Hrushovski,L)

For any countable, complete theory T with an infinite model, the uncountable spectrum $\aleph_\alpha \mapsto I(T,\aleph_\alpha)$ $(\alpha > 0)$ is the minimum of the map $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$ and one of the following maps:

1. $2^{\aleph_\alpha}$;
2. $\beth_{d+1}(|\alpha+\omega|)$ *for some* d, $\omega \leq d < \omega_1$;
3. $\beth_{d-1}(|\alpha+\omega|^{2^{\aleph_0}})$ *for some* d, $0 < d < \omega$;
4. $\beth_{d-1}(|\alpha+\omega|^{\aleph_0}+\beth_2)$ *for some* d, $0 < d < \omega$;
5. $\beth_{d-1}(|\alpha+\omega|+\beth_2)$, *for some* d, $0 < d < \omega$;
6. $\beth_{d-1}(|\alpha+\omega|^{\aleph_0})$, *for some* d, $0 < d < \omega$;
7. $\beth_{d-1}(|\alpha+\omega|+2^{\aleph_0})$, *for some* d, $1 < d < \omega$;
8. $\beth_{d-1}(|\alpha+\omega|)$, *for some* d, $0 < d < \omega$;
9. $\beth_{d-2}(|\alpha+\omega|^{|\alpha+1|})$, *for some* d, $1 < d < \omega$;
10. *identically* $\beth_2$;
11. $\begin{cases} |(\alpha+1)^n/\sim_G| - |\alpha^n/\sim_G| & \alpha < \omega; \\ |\alpha| & \alpha \geq \omega; \end{cases}$ *for some* $1 < n < \omega$ *and some group* $G \leq Sym(n)$
12. *identically 1.*

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A weight one extension N/M has depth 0 if any non-algebraic $q \in S(N)$ is nonorthogonal to M.

Theorem (Shelah)

If N/M is nonorthogonal to a nontrivial type, then N/M has depth 0.

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- There is a regular $p \in S(M)$ realized in N, say by a;
- N is dominated by a over M; and
- N is minimal over Ma.

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If both answers are YES, then N/M is 'ω-stable-like'.

Analyze this via the 'usual trichotomy' of regular types:

$$\begin{cases} p \text{ non-locally modular} & \text{`geometric'} \\ p \text{ locally modular, nontrivial} & \text{`linear'} \\ p \text{ trivial} & \end{cases}$$

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Corollary

If N/M is weight 1, non-orthogonal to a non-locally modular regular type, then N is prime and minimal over Ma for any $a \in N$ such that $\mathrm{tp}(a/M)$ is regular.

N/M non-locally modular

A generalization:

Theorem

If a stationary $q \in S(A)$ is p-semiregular (i.e., q domination equivalent to $p^{(n)}$ for some n) then q is strongly p-semiregular i.e., there is a formula $\theta \in q$ such that for any $B \supseteq A$, any $r \in S(B)$ containing θ, EITHER $w_p(r) < n$ OR r is the nonforking extension of q to $S(B)$.

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Corollary

If G is a p-semiregular group with p non-locally modular, then G is connected by finite (i.e., the connected component G^0 has finite index in G).

N/M nonorthogonal to a locally modular, nontrivial regular type p

In this case, there is a definable group controlling p.
Suppose G is an M-definable group, whose generics are locally modular and regular, $\not\perp p$. Then:

- $G_0 = \{g \in G : w_p(g/M) = 0\}$ is a subgroup of G (typically $\vee$-definable)

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- $G_0 = \{g \in G : w_p(g/M) = 0\}$ is a subgroup of G (typically $\vee$-definable)
- G is abelian (Poizat)
- There is a division ring E of $Cl_p(M)$-definable quasi-endomorphisms $S \subseteq G \times G$ of p-weight 1 that describes forking on $G^0 \setminus G_1$ (where $G_1 = G^0 \cap G_0$) i.e., Each S gives rise to an endomorphism $f_S : G^0/G_1 \to G^0/G_1$ and

$$\{g_1, \ldots, g_n\} \text{ are forking independent} \Leftrightarrow$$
$$\{g_1 + G_1, \ldots, g_n + G_1\} \text{ are } E\text{-linearly independent}$$

Another dichotomy

Definition

A locally modular, regular type p is limited if each endomorphism f_S is represented by an M-definable $S \subseteq G \times G$.

Note: Every minimal locally modular regular type is limited.

Example

E any division ring, $L_E = \{+, 0, \cdot_e\}_{e \in E}$, V any (infinite) E-vector space. Then $p(x) = \{x \neq 0\}$ is regular, locally modular and limited. The quasi-endomorphism ring is precisely E.

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Example

A two-sorted structure with sorts F and V.
$L = \{F, V, +_F, \cdot_F, 0_F, 1_F, 0_V, +_V, g\}$. F is a field imposing an F-vector space structure on V via g, i.e., $g(f, v)$ is scalar multiplication by f. The type $p(x) = \{V(x), x \neq 0\}$ is regular and locally modular, but not limited. The endomorphism ring E is isomorphic to F, and is represented by $g(f, \cdot)$ for $f \in F$. Note that $F \subseteq Cl_p(\emptyset)$.

Unlimited, locally modular types

Theorem (Loveys)

(T stable) Let G be an M-definable group, whose principal generic p is regular, locally modular, but not limited. Then there is an M-definable subgroup $H \subseteq G$ of finite index, and a $Cl_p(M)$-definable subgroup $K \subseteq H$ of p-weight 0 such that H/K is connected.

Corollary

If $p \in S(M)$ is regular, locally modular, but not limited, then for any $a \models p$, there is $a' \in \mathrm{acl}(Ma) \setminus M$ such that a'/M is strongly regular.

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In our context:

Corollary

If N/M is weight 1, nonorthogonal to a locally modular, not limited regular type, then there is $a \in N \setminus M$ such that $\mathrm{tp}(a/M)$ is strongly regular, and N is prime and minimal over Ma.

On the other hand...

Theorem

Suppose G is an M-definable group with limited, locally modular, regular generic types. Then the division ring E of quasi-endomorphisms describes forking on all of $G \setminus G_0$: Each quasi-endomorphism S induces an endomorphism $f_S^* : G/G_0 \to G/G_0$ and on $G \setminus G_0$,

$$\{g_1, \ldots, g_n\} \text{ are forking independent} \Leftrightarrow$$
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Theorem

If $Th(M)$ is classifiable and G is an M-definable group with limited, locally modular, regular generic types, then the expansion $M^* = (M, \ldots, G_0)$ formed by adding a predicate for the non-generic elements of G remains classifiable.

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BUT...

Theorem (MISLEADING!)

There is $a \in N$, the generic of a group, such that N is prime and minimal over Ma.

Example

V a vector space over F_2, with a family $\{V_n\}_{n\in\omega}$ of independent subspaces, each of codimension 1. Then there are no strongly regular types, but if V^* is a weight 1 extension of V, then $V^* = \mathrm{acl}(Va)$ for ANY $a \in V^* \setminus V$.

Look at covers of V:

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Example

Suppose we have the same V (with or without the subspaces), and an infinite set X. Consider the two-sorted structure $(V, X \times V, 0_V, +_V, \pi)$ where $\pi(x, v) = v$ for all $x \in X$. Then the fibers above each $v \in V$ are orthogonal, contradicting NDOP (prime models are not minimal).

To maintain minimality, we need a linkage between the fibers that is controlled by an M-definable set W.

Suppose that $0 \to W \to (W \times V) \to V \to 0$ is an exact sequence of vector spaces over F_2.

Code this as a three sorted structure with sorts $W \times V$, V and W, along with the embedding of W into $W \times V$ and the projection $\pi : W \times V \to V$. Endow W with independent subspaces $\{W_n\}$, each of finite index.

Suppose $M \subseteq N$ with $(b, a) \in N \setminus M$.

THEN: The element a is the generic of a group (namely V), but N is not prime over Ma.

However, the element (b, a) is the generic of a larger group $(W \times V)$ and N is prime over $M \cup \{(b, a)\}$.

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Theorem (almost true)

If N/M nonorthogonal to a limited, locally modular, regular type, then N is prime and minimal over Ma, where a is a $*$-definable element, and is generic for a projective limit of M-definable abelian groups.

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In many cases, e.g., if T has finite U-rank, then the above IS a theorem.

Theorem

N/M nonorthogonal to a limited, locally modular, regular type. Let T^* be the expansion of T formed by adding a predicate for G_0, the group of non-generics for every M-definable group G with regular, locally modular, limited generics. Let $\overline{G}$ be a maximal, projective system of M^*-definable (in T^*) groups, each with locally modular, limited, generics, and let $a \in N$ be a $**$-definable generic for $\overline{G}$. Then N is prime and minimal over Ma.

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- Enumerate $N \setminus Ma$ in "semiregular batches" $\langle C_\alpha \rangle$ i.e.,
  - for any finite $d \in C_\alpha$, $d/Ma\{C_\beta : \beta < \alpha\})$ is p_α-semiregular for some regular type p_α
  - Since N/M has depth 0, we may assume each $p_\alpha \in S(M)$
  - The essential U-rank $E(p_\alpha)$ =smallest R^∞-rank of a formula nonorthogonal to p_α is nondecreasing.
  - Each C_α is a maximal such subset of N.

Suppose we have shown that $C^*_\alpha := Ma \cup \bigcup\{C_\beta : \beta < \alpha\}$ is atomic over Ma and concentrate on $\text{tp}(d/C^*_\alpha)$, which is p_α-semiregular for some $p_\alpha \in S(M)$.

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• Since N/M is weight 1, d/C_α^* is almost orthogonal to p_α, but is not orthogonal to p_α.

THUS, p_α is nontrivial, and moreover there is an M-definable p_α-semiregular group H for which d/C_α^* is the type of a torsor.

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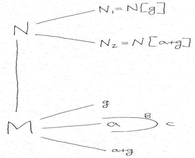
In this case, we prove a pair of group covering theorems.

A variant on V-domination:

Definition

Suppose $B \supseteq Ma$ is countable and atomic. The type c/B is *G-dominated* if, for all $g \in \overline{G}$ generic, for all N independent from Bcg over M, for all N_1 dominated by g over N, and all N_2 dominated by $a+g$ over N, we have $\mathrm{tp}(c/BN_1N_2)$ does not fork over B.



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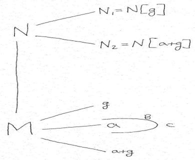

Since T classifiable, c/B G-dominated implies c/B isolated.

Main point: If d/C^*_α is p_α-semiregular for some limited, locally modular p_α, but is not G-dominated, then there is an M-definable group G' properly projecting onto G, contradicting the maximality of the projective system $\overline{G}$.

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• If no element of p_α-weight zero is nonorthogonal to $\mathrm{tp}(a/M)$ then this is akin to the "usual" group existence theorem for locally modular types. If T has finite U-rank then we are always in this case.

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• In the second construction, we require that the subgroup H_0 of p_α-nongeneric elements be M-definable. Here is where the expansion of the language is used.

There seem to be parallels with Buecher's proof of Vaught's conjecture for theories of finite U-rank.

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Conjecture

There is $a \in N$ and b, the generic of a $**$-definable projective system of groups such that N is prime and minimal over Mab.